VECTOR ALGEBRA

IMPORTANT POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a **vector**. It is denoted by a directed line segment.
- Two or more vectors which are parallel to same line are called **collinear vectors**.
- Position vector of a point P(a, b, c) w.r.t. origin (0, 0, 0) is denoted by $\overrightarrow{OP}$, where $\overrightarrow{OP}$ = a$\hat{\imath} + b\hat{\jmath} + c\hat{k}$ and l $\overrightarrow{OP}$ l = $\sqrt{a^2 + b^2 + c^2}$.
- If A(x_1, y_1, z_1) and B (x_2, y_2, z_2) be any two points in space, then
  $\overrightarrow{AB}$ = (x_2 - x_1)$\hat{\imath}$ + (y_2 - y_1)$\hat{\jmath}$ + (z_2 - z_1)$\hat{k}$ and
  l$\overrightarrow{AB}$l = $\sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of triangle taken in opposite order. This is called **triangle law of addition of vectors.**
- If $\vec{a}$ is any vector and λ is a scaler, then $\lambda\vec{a}$ is a vector collinear with $\vec{a}$ and l$\lambda\vec{a}$l = lλl l$\vec{a}$l.
- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\vec{b}$, where λ is some scaler.
- Any vector $\vec{a}$ can be written as $\vec{a}$ = l$\vec{a}$l $\hat{a}$, where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.
- If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B, and C is any point which divides $\overrightarrow{AB}$ in the ratio m : n internally then position vector $\vec{c}$ of point C is given as $\vec{c} = \frac{m\vec{b}+ n\vec{a}}{m+n}$. If C divides $\overrightarrow{AB}$ in ratio m : n externally, then $\vec{c} = \frac{m\vec{b} - n\vec{a}}{m- n}$.
- The angles α, β *and* γ made by $\vec{r}$ = a$\hat{\imath}$+b$\hat{\jmath}$ + c$\hat{k}$ with positive direction of x, y and z-axis are called **direction angles** and cosines of these angles are called **direction cosines** of $\vec{r}$ denoted as l = cosα, m = cosβ, n = cosγ. Also l = $\frac{a}{l\vec{r}l}$, m = $\frac{b}{l\vec{r}l}$, n = $\frac{c}{l\vec{r}l}$ & $l^2 + m^2 + n^2 = 1$
- The numbers a, b, c proportional to l, m, n are called **direction ratios.**
- **Scaler product** of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \,.\, \vec{b}$ & defined as $\vec{a} \,.\, \vec{b}$ = l$\vec{a}$ ll $\vec{b}$lcosθ where θ is the angle between $\vec{a}$ and $\vec{b}$. ($0 \leq \theta \leq \pi$).
- $\vec{a}.\vec{b} = 0$ if and only if $\vec{a} = \vec{0}, \vec{b} = \vec{0}$ *or* $\vec{a}$ is perpendicular to $\vec{b}$.
- $\vec{a}.\vec{a}$ = l$\vec{a}$l^2, so $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k}$ = 1.
- If $\vec{a}$ = $a_1\hat{\imath}$+ $a_2\hat{\jmath}$ +$a_3\hat{k}$, $\vec{b}$ = $b_1\hat{\imath}$+ $b_2\hat{\jmath}$ +$b_3\hat{k}$, then $\vec{a}.\vec{b}$ = $a_1b_1 + a_2b_2 + a_3b_3$.
- **Cross product (Vector product)** of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}$ x $\vec{b}$ & defined as $\vec{a}$ x $\vec{b}$ = l$\vec{a}$ ll $\vec{b}$l sinθ $\hat{n}$,where θ is the angle between $\vec{a}$ and $\vec{b}$. ($0 \leq \theta \leq \pi$) and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system.
- $\vec{a}$ x $\vec{b}$ = $\vec{0}$ iff $\vec{a} = \vec{0}, \vec{b} = \vec{0}$ *or* $\vec{a}$ is parallel to $\vec{b}$.

- Unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ = $\pm\ \frac{(\vec{a}\ X\ \vec{b})}{l\ \vec{a}\ X\ \vec{b}\ l}$
- **l$\vec{a}$ x $\vec{b}$ l** is the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.
- $\frac{1}{2}$ **l $\vec{a}$ x $\vec{b}$ l** is the area of parallelogram whose diagonals are $\vec{a}$ and $\vec{b}$.

ASSIGNMENT

1. If $\vec{a}, \vec{b}$ are the position vectors of the points (1, -1), (-2, m), find the value of m for which $\vec{a}$ & $\vec{b}$ are collinear.
2. If a vector makes angles α, β, γ with OX, OY and OZ respectively, prove that $sin^2\alpha + \sin^2\beta\ +\ sin^2\gamma = 2$.
3. Find the direction cosines of a vector $\vec{r}$ which is equally inclined with OX, OY and OZ. If l $\vec{r}$ l is given, find the total number of such vectors.
4. A vector $\vec{r}$ is inclined at equal angles to OX, OY and OZ. If the magnitude of $\vec{r}$ is 6 units, find $\vec{r}$.
5. A vector $\vec{r}$ has length 21 and d. r.s 2, -3, 6. Find the direction cosines and components of $\vec{r}$, given that it makes an acute angle with x axis.
6. Find the angles at which the vector $2\hat{\imath}$ -$\hat{\jmath}$ + $2\hat{k}$ is inclined to each of the coordinate axes.
7. For any vector $\vec{r}$, prove that $\vec{r}$ = ($\vec{r}.\ \hat{\imath}$)$\hat{\imath}$ + ($\vec{r}.\ \hat{\jmath}$)$\hat{\jmath}$ + ($\vec{r}.\hat{k}$)$\hat{k}$.
8. Find the value of p for which the vectors $\vec{a}\ = 3\hat{\imath}+2\hat{\jmath}+9\hat{k}$ and $\vec{b}\ =\ \hat{\imath}+ p\hat{\jmath}+3\hat{k}$ are
   i. Perpendicular ii. Parallel
9. If $\hat{a}$ & $\hat{b}$ are unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2} = \frac{1}{2}$ l $\hat{a} - \hat{b}$ l.
10. If $\vec{a}$ makes equal angles with $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$ and has magnitude 3, then prove that the angle between $\vec{a}$ and each of $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$ is $\cos^{-1}(1/\sqrt{3})$.
11. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, l$\vec{a}$l = 3, l$\vec{b}$l = 5 and l$\vec{c}$l = 7, find the angle between $\vec{a}$ & b.
12. Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{\imath}$ -$\hat{\jmath}$ +$3\hat{k}$ & $-2\hat{\imath}+\hat{\jmath}-2\hat{k}$.
13. Find a unit vector perpendicular to the plane ABC where A, B, C are the points (3, -1, 2), (1, -1, -3), (4, -3, 1) respectively.
14. Show that area of a parallelogram having diagonals $3\hat{\imath}+\hat{\jmath} - 2\hat{k}$ and $\hat{\imath}-3\hat{\jmath} + 4\hat{k}$ is $5\sqrt{3}$.
15. Show that $(\vec{a} \text{ x } \vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$
16. Prove that the points A, B & C with position vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ respectively are collinear if and only if $\vec{a}$ x $\vec{b}$ + $\vec{b}$ x $\vec{c}$ + $\vec{c}$ x $\vec{a}$ = $\vec{0}$.
17. If $\vec{a}$ x $\vec{b}$ = $\vec{c}$ x $\vec{d}$ and $\vec{a}$ x $\vec{c}$ = $\vec{b}$ x $\vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}, \vec{b} \neq \vec{c}$
18. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be unit vectors such that $\vec{a}$. $\vec{b}$ = $\vec{a}$. $\vec{c}$ = 0 and the angle between $\vec{b}$ & $\vec{c}$ is $\frac{\pi}{6}$, prove that $\vec{a}\ =\ \pm 2\ (\vec{b}$ x $\vec{c})$

19. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.
20. If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$, then show that $\vec{b} = \vec{c}$.
21. Show that the vectors $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ are collinear.
22. Find λ, if $(2\hat{\imath} + 6\hat{\jmath} + 14\hat{k}) \times (\hat{\imath} - \lambda\hat{\jmath} + 7\hat{k}) = \vec{0}$.
23. Let $\vec{a} = \hat{\imath} + 4\hat{\jmath} + 2\hat{k}$, $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$ and $\vec{c} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$. Find a vector $\vec{p}$ which is is perpendicular to both $\vec{a}$ & $\vec{b}$ and $\vec{p} \cdot \vec{c} = 18$.
24. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}| = 5$, $|\vec{b}| = 12$ and $|\vec{c}| = 13$, and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.
25. Find the position vector of a point R which divides the line joining the two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ respectively, externally in the ratio 1 : 2. Also, show that P is the midpoint of the line segment RQ.
26. Find a unit vector perpendicular to each of the vectors $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.
27. If $|\vec{a}| = 13$, $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 60$, then find $|\vec{a} \times \vec{b}|$.
28. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} \cdot \vec{b}$.
29. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{\jmath} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$.
30. Express the vector $\vec{a} = 5\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{\imath} + \hat{k}$ and other is perpendicular to $\vec{b}$.
31. If the vertices A, B, C of triangle ABC have position vectors (1, 2, 3) (-1, 0, 0), (0, 1, 2) respectively, what is the magnitude of angle ABC?
32. If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$.
33. Show that vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ is equally inclined to the axes.
34. Show that three points $\vec{a} - \vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear.
35. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

Answer Key

1. m = 2 3. $\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}$; 8 ways 4. $\vec{r} = 2\sqrt{3}(\pm\hat{\imath} \pm\hat{\jmath} \pm\hat{k})$

5. 2/7, -3/7, 6/7 ; $\vec{r} = 6\hat{\imath} - 9\hat{\jmath} + 18\hat{k}$ 6. $\alpha = \cos^{-1} 2/3$, $\beta = \cos^{-1} -1/3$, $\gamma = \cos^{-1} 2/3$

8. i. p = -15 ii. p = 2/3 11. $\pi/3$ 12. $-3\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$ 13. $\frac{1}{\sqrt{165}}(-10\hat{\imath} - 7\hat{\jmath} + 4\hat{k})$

22. -3 23. $\pm(64\hat{\imath} - 2\hat{\jmath} - 28\hat{k})$ 24. -169 25. $3\vec{a} + 5\vec{b}$ 26. $\frac{1}{3}(2\hat{\imath} - 2\hat{\jmath} - \hat{k})$

27. 25 28. 6 29. $\frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$ 30. $6\hat{\imath} + 2\hat{k}$, $-\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

31. $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ 32. 3