TOPIC 10
THREE DIMENSIONAL GEOMETRY
SCHEMATIC DIAGRAM

Topic	Concept	Degree of importance	Refrence NCERT Text Book Edition 2007
Three Dimensional Geometry	(i) Direction Ratios and Direction Cosines	*	Ex No 2 Pg -466 Ex No 5 Pg – 467 Ex No 14 Pg - 480
	(ii)Cartesian and Vector equation of a line in space & conversion of one into another form	**	Ex No 8 Pg -470 Q N. 6, 7, - Pg 477 QN 9 – Pg 478
	(iii)Co-planer and skew lines	*	Ex No 29 Pg -496
	(iv) Shortest distance between two lines	***	Ex No 12 Pg -476 Q N. 16, 17 - Pg 478
	(v) Cartesian and Vector equation of a plane in space & conversion of one into another form	**	Ex No 17 Pg -482 Ex No 18 Pg – 484 Ex No 19 Pg – 485 Ex No 27 Pg – 495 Q N. 19, 20 - Pg 499
	(vi) Angle Between (iv) Two lines (v) Two planes (vi) Line & plane	 * * **	Ex No 9 Pg -472 Q N. 11 - Pg 478 Ex No 26 Pg – 494 Q N. 12 - Pg 494 Ex No 25 Pg - 492
	(vii) Distance of a point from a plane	**	Q No 18 Pg -499 Q No 14 Pg – 494
	(viii)Distance measures parallel to plane and parallel to line	**	
	(ix)Equation of a plane through the intersection of two planes	***	Q No 10 Pg -493
	(x) Foot of perpendicular and image with respect to a line and plane	**	Ex. N 16 Pg 481

SOME IMPORTANT RESULTS/CONCEPTS

** Direction cosines and direction ratios :

If a line makes angles α, β and γ with x, y and z axes respectively the $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the direction cosines denoted by l, m, n respectively and $l^2 + m^2 + n^2 = 1$

Any three numbers proportional to direction cosines are direction ratios denoted by a, b, c

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c} \qquad l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

* Direction ratios of a line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $x_2-x_1,\ y_2-y_1,\ z_2-z_1$
* Angle between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is given by

$$\cos\theta = l_1l_2+m_1m_2+n_1n_2=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}$$

* For parallel lines $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ and

for perpendicular lines $a_1a_2+b_1b_2+c_1c_2=0$ or $l_1l_2+m_1m_2+n_1n_2=0$

** STRAIGHTLINE:

* Equation of line passing through a point (x_1, y_1, z_1) with direction cosines a, b, c: $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$
* Equation of line passing through a point (x_1, y_1, z_1) and parallel to the line: $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

* Equation of line passing through two point (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$
* Equation of line (Vector form)

Equation of line passing through a point $\vec{a}$ and in the direction of $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$

* Equation of line passing through two points $\vec{a}$ & $\vec{b}$ and in the direction of $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$
* Shortest distance between two skew lines : if lines are $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$

then Shortest distance $=\frac{\left(\vec{a_2}-\vec{a_1}\right)\left(\vec{b_1}\times\vec{b_2}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}$; $\vec{b_1}\times\vec{b_2}\neq 0$

$$\frac{\left|\left(\vec{a_2}-\vec{a_1}\right)\times\vec{b_1}\right|}{\left|\vec{b_1}\right|} \quad ;\vec{b_1}\times\vec{b_2}=0$$

**PLANE:

* Equation of plane is $ax + by + cz + d = 0$ where a, b & c are direction ratios of normal to the plane
* Equation of plane passing through a point (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$
* Equation of plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a, b, c are intercepts on the axes
* Equation of plane in normal form $lx+my+nz=p$ where l, m, n are direction cosines of normal to the plane p is length of perpendicular form origin to the plane

* Equation of plane passing through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and $(x_3, y_3, z\ \ )$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

* Equation of plane passing through two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and perpendicular to the plane

$a_1x + b_1y + c_1z + d_1 = 0$ or parralal to the line $\dfrac{x - \alpha_1}{a_1} = \dfrac{y - \beta_1}{b_1} = \dfrac{z - \gamma_1}{c_1}$ is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

* Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to the

planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z \ + d_2 = 0$ or parralal to the lines $\dfrac{x - \alpha_1}{a_1} = \dfrac{y - \beta_1}{b_1} = \dfrac{z - \gamma_1}{c_1}$

and $\dfrac{x - \alpha_2}{a_2} = \dfrac{y - \beta_2}{b_2} = \dfrac{z - \gamma_2}{c_2}$ is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

* Equation of plane contaning the line $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and passing through the point (x_2, y_2, z_2) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

* Condition for coplaner lines : $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ are coplaner if

$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$ and equation of common plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

* Equation of plane passing through the intersection of two planes $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ is $(a_1x + b_1y + c_1z) + \lambda(a_2x + b_2y + c_2z) = 0$

* Perpendicular distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is $\dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$

* Distance between two parallel planes $ax + by + cz + d_1 = 0$, $ax + by + cz + d_2 = 0$ is $\dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

ASSIGNMENTS

(i)Direction Ratios and Direction Cosines

LEVEL-I

1. Write the direction-cosines of the line joining the points (1,0,0) and (0,1,1) **[CBSE 2011]**

2.Find the direction cosines of the line passing through the following points (-2,4,-5), (1,2,3).

3.Write the direction cosines of a line equally inclined to the three coordinate axes

LEVEL-II

1.Write the direction cosines of a line parallel to the line $\frac{3-x}{3} = \frac{y+2}{-2} = \frac{z+2}{6}$.

2.Write the direction ratios of a line parallel to the line $\frac{5-x}{3} = \frac{y+7}{-2} = \frac{z+2}{6}$.

3. If the equation of a line AB $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+6}{3}$ Find the direction cosine.

4. Find the direction cosines of a line, passing through origin and lying in the first octant, making equal angles with the three coordinate axis.

(ii) Cartesian and Vector equation of a line in space & conversion of one into another form

LEVEL-I

1.Write the vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$. **[CBSE 2011]**

2. Write the equation of a line parallel to the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+5}{6}$ and passing through the point(1,2,3).

3.Express the equation of the plane $\vec{r} = (\hat{\imath} - 2\hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$ in the Cartesian form.

4.Express the equation of the plane $\vec{r} \cdot (2\hat{\imath} - 3\hat{\jmath} + \hat{k}) + 4 = 0$ in the Cartesian form.

(iii) Co-planer and skew lines

LEVEL-II

1.Find whether the lines $\vec{r} = (\hat{\imath} - \hat{\jmath} - \hat{k}) + \lambda(2\hat{\imath} + \hat{\jmath})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath}) + \mu(\hat{\imath} + \hat{\jmath} - \hat{k})$ intersect or not. If intersecting , find their point of intersection.

2.Show that the four points (0,-1,-1) , (4,5,1) , (3,9,4) and (-4,4,4,) are coplanar. Also, find the equation of the plane containing them.

3.Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find their point of intersection.

LEVEL-III

1. Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar. Also find the equation of the plane.

2. The points A(4,5,10) , B(2,3,4) and C(1,2,-1) are three vertices of a parallelogram ABCD. Find

the vector equation of the sides AB and BC and also find the coordinates

3.Find the equations of the line which intersects the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and passes through the point (1,1,1).

4. Show that The four points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (– 4, 4, 4) are coplanar and find the equation of the common plane .

(iv) Shortest distance between two lines

LEVEL-II

1. Find the shortest distance between the lines l_1 and l_2 given by the following:

(a) $l_1 : \frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-1}{1}$ $\quad$ $l_2 : \frac{x-2}{2}=\frac{y+1}{1}=\frac{z+1}{2}$

(b) $\vec{r}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\lambda(\hat{\imath}-3\hat{\jmath}+2\hat{k})$

$\vec{r}=(4\hat{\imath}+2\mu)\hat{\imath}+(5+3\mu)\hat{\jmath}+(6+\mu)\hat{k}$.

2. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Find their point of intersection.

3.. Find the shortest distance between the lines

$\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$, and $\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\mu(4\hat{i}-2\hat{j}+2\hat{k})$

4.Find the shortest distance between the lines

$\vec{r}=(1-t)\hat{\imath}+(t-2)\hat{\jmath}+(3-t)\hat{k}$ and $\vec{r}=(s+1)\hat{\imath}+(2s-1)\hat{\jmath}+(2s+1)\hat{k}$[**CBSE 2011]**

5. Find the distance between the parallel planes x + y – z = -4 and 2x + 2y – 2z + 10 = 0.

6. Find the vector equation of the line parallel to the line $\frac{x-1}{5}=\frac{3-y}{2}=\frac{z+1}{4}$ and passing through (3,0,-4). Also, find the distance between these two lines.

(v) Cartesian and Vector equation of a plane in space & conversion of one into another form

LEVEL I

1.Find the equation of a plane passing through the origin and perpendicular to x-axis

2.Find the equation of plane with intercepts 2, 3, 4 on the x ,y, z –axis respectively.

3.Find the direction cosines of the unit vector perpendicular to the plane

$\vec{r}\cdot(6\hat{\imath}-3\hat{\jmath}-2\hat{k})+1=0$ passing through the origin.

4.Find the Cartesian equation of the following planes:

(a) $r\cdot(\hat{\imath}+\hat{\jmath}-\hat{k})=2$ $\quad$ (b) $r\cdot(2\hat{\imath}+3\hat{\jmath}-4\hat{k})=1$

LEVEL II

1. Find the vector and cartesian equations of the plane which passes through the point (5, 2, – 4) and perpendicular to the line with direction ratios 2, 3, – 1.
2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector 3 iˆ + 5 ˆj - 6 kˆ.
3.Find the vector and cartesian equations of the planes that passes through the point (1, 0, – 2)and the normal to the plane is iˆ + ˆj - kˆ.

(vi) Angle Between(i)Two lines (ii)Two planes (iii)Line & plane

LEVEL-I

1. Find the angle between the lines whose direction ratios are (1, 1, 2) and ($\sqrt{3}$–1 , –$\sqrt{3}$ –1 , 4).

2.Find the angle between line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane 3x + 4y + z + 5 = 0.

3.Find the value of λ such that the line $\frac{x-2}{9} = \frac{y-1}{\lambda} = \frac{z+3}{-6}$ is perpendicular to the plane 3x – y – 2z = 7.

4.Find the angle between the planes whose vector equations are
r· (2 iˆ + 2 ˆj - 3 kˆ) = 5 and r · (3 iˆ - 3 ˆj + 5 kˆ) = 3

5.Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane 10 x + 2y – 11 z = 3.

LEVEL-II

1.Find the value of p, such that the lines $\frac{x}{1} = \frac{y}{3} = \frac{z}{2p}$ and $\frac{x}{-3} = \frac{y}{5} = \frac{z}{2}$ are perpendicular to each other.

2. A line makes angles α, β, γ, δ with the four diagonals of a cube, Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

(vii) Distance of a point from a plane

LEVELI

1.Write the distance of plane 2x– y + 2z + 1 = 0 from the origins.
2.Find the point through which the line 2x = 3y = 4z passes.
3. Find the distance of a point $(2, 5, -3)$from the plane $\mathrm{r} \cdot (6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = 4$
4. Find the distance of the following plane from origin: 2x – y + 2z + 1 = 0

5.Find the distance of the point (a,b,c) from x-axis

LEVELII

1..Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point P(1,3,3).

2.Find the distance of the point (3,4,5) from the plane x + y + z =2 measured parallel to the line 2x = y = z.

3. Find the distance between the point P(6, 5, 9) and the plane determinedby the points A (3, – 1, 2), B (5, 2, 4) and C(– 1, – 1, 6).

4.Find the distance of the point $(-1,-5,-10)$ from the point of intersection of theline

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right) \text{ and the plane } \vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$$

[**CBSE2011**]

LEVEL III

1.Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (1,3,4) from the plane 2x – y + z + 3 = 0. Find also, the image of the point in the plane.

2.Find the distance of the point P(6,5,9) from the plane determined by the points A(3,-1,2) , B(5,2,4) and C(-1,-1,6).

3.Find the equation of the plane containing the lines $\vec{r} = \hat{\imath} + \hat{\jmath} + \lambda(\hat{\imath} + 2\hat{\jmath} - \hat{k})$ and $\vec{r} = \hat{\imath} + \hat{\jmath} + \mu(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$.Find the distance of this plane from origin and also from the point (1,1,1).

(viii) Equation of a plane through the intersection of two planes

LEVELII

1.Find the equation of plane passing through the point (1,2,1) and perpendicular to the line joining the points (1,4,2) and (2,3,5). Also find the perpendicular distance of the plane from the origin.

2.Find the equation of the plane which is perpendicular to the plane 5x + 3y + 6z +8 = 0 and which contains the line of intersection of the planes x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0.

3.Find the equation of the plane that contains the point (1,-1,2) and is perpendicular to each of the planes 2x + 3y -2 z = 5 and x + 2y – 3z = 8.

LEVEL-III

1.Find the equation of the plane passing through the point (1,1,1) and containing the line $\vec{r} = (-3\hat{\imath} + \hat{\jmath} + 5\hat{k}) + \lambda(3\hat{\imath} - \hat{\jmath} - 5\hat{k})$. Also, show that the plane contains the line $\vec{r} = (-\hat{\imath} + 2\hat{\jmath} + 5\hat{k}) + \lambda(\hat{\imath} - 2\hat{\jmath} - 5\hat{k})$.

2.Find the equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes x + 2y + 3z – 7 = 0 and 2x – 3y + 4z = 0.

3.Find the Cartesian equation of the plane passing through the points A(0,0,0) and B(3,-1,2) and parallel to the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ `

4. Find the equation of the perpendicular drawn from the point P(2,4,-1) to the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}.$$

(ix)Foot of perpendicular and image with respect to a line and plane

LEVEL II

1. Find the coordinates of the point where the line through (3,-4,-5) and (2,-3,1) crosses the plane determined by points A(1,2,3) , B(2,2,1) and C(-1,3,6).

2. Find the foot of the perpendicular from P(1,2,3) on the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$. Also, obtain the equation of the plane containing the line and the point (1,2,3).

3.Prove that the image of the point (3,-2,1) in the plane $3x - y + 4z = 2$ lies on the plane, $x + y + z + 4 = 0$.

LEVEL-III

1.Find the foot of perpendicular drawn from the point A(1, 0, 3) to the joint of the points B(4, 7, 1) and C(3, 5, 3).

2. Find the image of the point (1, –2, 1) in the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z+3}{2}$.

3. The foot of the perpendicular from the origin to the plane is (12, – 4, 3). Find the equation of the plane

4. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point P(3,2,1) from the plane $2x - y + z + 1 = 0$. Find also, the image of the point in the plane.

Questions for self evaluation

1. Find the equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

2. Find the vector equation of a line joining the points with position vectors $\hat{i} - 2\hat{j} - 3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i} - \hat{j} + 4\hat{k}$, and $2\hat{i} + \hat{j} + 2\hat{k}$. Also find the cartesian equivalent of this equation.

3. Find the foot of perpendicular drawn from the point A(1, 0, 3) to the joint of the points B(4, 7, 1) and C(3, 5, 3).

4. Find the shortest distance between the lines

$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$, and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(4\hat{i} - 2\hat{j} + 2\hat{k})$

5.Find the image of the point (1, –2, 1) in the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z+3}{2}$.

6. Show that the four points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (– 4, 4, 4) are coplanar and find the equation of the common plane .

7. The foot of the perpendicular from the origin to the plane is (12, – 4, 3). Find the equation of the plane.

8. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find their point of intersection.

9. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$