TOPIC 3
MATRICES & DETERMINANTS
SCHEMATIC DIAGRAM

Topic	Concepts	Degree of importance	References NCERT Text Book XI Ed. 2007
Matrices & Determinants	(i) Order, Addition, Multiplication and transpose of matrices	***	Ex 3.1 –Q.No 4,6 Ex 3.2 –Q.No 7,9,13,17,18 Ex 3.3 –Q.No 10
	(ii) Cofactors &Adjoint of a matrix	**	Ex 4.4 –Q.No 5 Ex 4.5 –Q.No 12,13,17,18
	(iii)Inverse of a matrix & applications	***	Ex 4.6 –Q.No 15,16 Example –29,30,32 ,33 MiscEx 4–Q.No 4,5,8,12,15
	(iv)To find difference between $\|A\|$, $\|adj\ A\|$, $\|kA\|$, $\|A.adjA\|$	*	Ex 4.1 –Q.No 3,4,7,8
	(v) Properties of Determinants	**	Ex 4.2–Q.No 11,12,13 Example –16,18

SOME IMPORTANT RESULTS/CONCEPTS

A matrix is a rectangular array of $m \times n$ numbers arranged in m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots\ldots a_{2n} \\ & \\ a_{m1} & a_{m2} \ldots\ldots\ldots a_{mn} \end{bmatrix}_{m\times n} \quad \text{OR } A = [a_{ij}]_{m\times n}, \text{ where } \mathbf{i} = 1, 2,\ldots, m ; \mathbf{j} = 1, 2,\ldots,n.$$

* **Row Matrix**: A matrix which has one row is called row matrix. $A = [a_{ij}]_{1\times n}$

* **Column Matrix** : A matrix which has one column is called column matrix. $A = [a_{ij}]_{m\times 1}$.

* **Square Matrix**: A matrix in which number of rows are equal to number of columns, is called a square matrix $A = [a_{ij}]_{m\times m}$

* **Diagonal Matrix** : A square matrix is called a Diagonal Matrix if all the elements, except the diagonal elements are zero. $A = [a_{ij}]_{n\times n}$, where $a_{ij} = 0$, $\mathbf{i} \neq j$.

$a_{ij} \neq 0$, $\mathbf{i} = j$.

* **Scalar Matrix**: A square matrix is called scalar matrix it all the elements, except diagonal elements are zero and diagonal elements are same non-zero quantity.

$A = [a_{ij}]_{n\times n}$, where $a_{ij} = 0$, $\mathbf{i} \neq j$.

$a_{ij} \neq \alpha$, $\mathbf{i} = j$.

* **Identity or Unit Matrix** : A square matrix in which all the non diagonal elements are zero and diagonal elements are unity is called identity or unit matrix.

*** Null Matrices** : A matrices in which all element are zero.
*** Equal Matrices** : Two matrices are said to be equal if they have same order and all their corresponding elements are equal.
* **Transpose of matrix** : If A is the given matrix, then the matrix obtained by interchanging the rows and columns is called the transpose of a matrix.\
* **Properties of Transpose** :
If A & B are matrices such that their sum & product are defined, then

(i). $\left(A^T\right)^T = A$ (ii). $(A+B)^T = A^T + B^T$ (iii). $\left(KA^T\right) = K.A^T$ where K is a scalar.
(iv). $(AB)^T = B^T A^T$ (v). $(ABC)^T = C^T B^T A^T$.

* **Symmetric Matrix** : A square matrix is said to be symmetric if A = A^T i.e. If $A = [a_{ij}]_{m\times m}$, then $a_{ij} = a_{ji}$ for all i, j. Also elements of the symmetric matrix are symmetric about the main diagonal
* **Skew symmetric Matrix** : A square matrix is said to be skew symmetric if A^T = -A.
If $A = [a_{ij}]_{m\times m}$, then $a_{ij} = -a_{ji}$ for all i, j.
***Singular matrix**:A square matrix 'A' of order 'n' is said to be singular, if | A| = 0.
*** Non -Singular matrix** : A square matrix 'A' of order 'n' is said to be non-singular, if | A| ≠ 0.

***Product of matrices:**

(i) If A & B are two matrices, then product AB is defined, if

Number of column of A = number of rows of B.

i.e. $A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times p}$ then AB = $AB = [C_{ik}]_{m\times p}$.

(ii) Product of matrices is not commutative. i.e. AB ≠ BA.

(iii) Product of matrices is associative. i.e A(BC) = (AB)C

(iv) Product of matrices is distributive over addition.

***Adjoint of matrix** :

If $A = [a_{ij}]$ be a n-square matrix then transpose of a matrix $[A_{ij}]$,
where A_{ij} is the cofactor of A_{ij} element of matrix A, is called the adjoint of A.

Adjoint of A = Adj. A = $[A_{ij}]^T$.

A(Adj.A) = (Adj. A)A = | A| I.

***Inverse of a matrix** :Inverse of a square matrix A exists, if A is non-singular or square matrix

A is said to be invertible and $A^{-1} = \frac{1}{|A|}$ Adj.A

***System of Linear Equations** :

$a_1x + b_1y + c_1z = d_1$.
$a_2x + b_2y + c_2z = d_2$.
$a_3x + b_3y + c_3z = d_3$.

$$\begin{bmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow A\,X = B \;\; \Rightarrow X = A^{-1}B \;\; ; \{\, |\,A\,| \neq 0\}.$$

***Criteria of Consistency**.

(i) If $|A| \neq 0$, then the system of equations is said to be consistent & has a unique solution.

(ii) If $|A| = 0$ and (adj. A)B = 0, then the system of equations is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and (adj. A)B $\neq 0$, then the system of equations is inconsistent and has no solution.

*** Determinant** :

To every square matrix we can assign a number called determinant

If $A = [a_{11}]$, det. $A = |\,A\,| = a_{11}$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $|A| = a_{11}a_{22} - a_{21}a_{12}$.

*** Properties** :

(i) The determinant of the square matrix A is unchanged when its rows and columns are interchanged.

(ii) The determinant of a square matrix obtained by interchanging two rows(or two columns) is negative of given determinant.

(iii) If two rows or two columns of a determinant are identical, value of the determinant is zero.

(iv) If all the elements of a row or column of a square matrix A are multiplied by a non-zero number k, then determinant of the new matrix is k times the determinant of A.

If elements of any one column(or row) are expressed as sum of two elements each, then determinant can be written as sum of two determinants.

Any two or more rows(or column) can be added or subtracted proportionally.

If A & B are square matrices of same order, then $|AB| = |A|\,|B|$

ASSIGNMENTS

(i). Order, Addition, Multiplication and transpose of matrices:

LEVEL I

1. If a matrix has 5 elements, what are the possible orders it can have? **[CBSE 2011]**

2. Construct a 3×2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j\,|$

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, then find A –2 B.

4. If $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$, write the order of AB and BA.

LEVEL II

1. For the following matrices A and B, verify $(AB)^T = B^TA^T$,

where $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1 \quad 2 \quad 1]$

2. Give example of matrices A & B such that AB = O, but BA $\neq$ O, where O is a zero matrix and

A, B are both non zero matrices.

3. If B is skew symmetric matrix, write whether the matrix (ABA^T) is Symmetric or skew symmetric.
4. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find a and b so that $A^2 + aI = bA$

LEVEL III

1. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find the value of $A^2 - 3A + 2I$

2. Express the matrix A as the sum of a symmetric and a skew symmetric matrix, where:

$$A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

3. If $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} a^n & \frac{b(a^n-1)}{a-1} \\ 0 & 1 \end{bmatrix}$, $n \epsilon N$

(ii) Cofactors &Adjoint of a matrix

LEVEL I

1. Find the co-factor of a_{12} in $A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$
2. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$

LEVEL II

Verify A(adjA) = (adjA) A = $|A|$I if

1. $A = \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(iii)Inverse of a Matrix & Applications

LEVEL I

1. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A **CBSE 2011**
2. If A is square matrix satisfying $A^2 = I$, then what is the inverse of A ?
3. For what value of k , the matrix $A = \begin{bmatrix} 2-k & 3 \\ -5 & 1 \end{bmatrix}$ is not invertible ?

LEVEL II

1. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = 0$. Hence find A^{-1}

2. If A, B, C are three non zero square matrices of same order, find the condition

on A such that AB = AC $\Rightarrow$ B = C.

3. Find the number of all possible matrices A of order 3 × 3 with each entry 0 or 1 and for which A $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions.

LEVEL III

1 If A = $\begin{bmatrix} 2 & 3 & 1 \\ -3 & 2 & 1 \\ 5 & -4 & -2 \end{bmatrix}$, find A^{-1} and hence solve the following system of equations:

2x – 3y + 5z = 11, 3x + 2y – 4z = - 5, x + y – 2z = - 3

2. Using matrices, solve the following system of equations:

a. x + 2y - 3z = - 4
2x + 3y + 2z = 2
3x - 3y – 4z = 11 [**CBSE 2011]**

b. 4x + 3y + 2z = 60
x + 2y + 3z = 45
6x + 2y + 3z = 70 [**CBSE 2011]**

3. Find the product AB, where A = $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and use it to solve the equations x – y = 3, 2x + 3y + 4z = 17, y + 2z = 7

4. Using matrices, solve the following system of equations:

$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4$

$\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$

5. Using elementary transformations, find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(iv)To Find The Difference Between $|A|, |adjA|, |kA|$

LEVEL I

1. Evaluate $\begin{vmatrix} cos15^{\circ} & sin15^{\circ} \\ sin75^{\circ} & cos75^{\circ} \end{vmatrix}$ **[CBSE 2011]**
2. What is the value of |3I| , where I is identity matrix of order 3?
3. If A is non singular matrix of order 3 and $|A|$ = 3, then find $|2A|$
4. For what valve of a, $\begin{bmatrix} 2a & -1 \\ -8 & 3 \end{bmatrix}$ is a singular matrix?

LEVEL II

1. If A is a square matrix of order 3 such that $|adjA|$ = 64, find $|A|$
2. If A is a non singular matrix of order 3 and $|A|$ = 7, then find $|adjA|$

LEVEL III

1. If $A = \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix}$ and $|A|^3 = 125$, then find a.
2. A square matrix A, of order 3, has $|A| = 5$, find $|A.adjA|$

(v).Properties of Determinants

LEVEL I

1. Find positive valve of x if $\begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}$

2. Evaluate $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

LEVEL II

1. Using properties of determinants, prove the following :

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

[CBSE 2012]

2. $$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

3. $$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

4. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$ **[CBSE 2012]**

LEVEL III

1. Using properties of determinants, solve the following for x :

a. $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ **[CBSE 2011]**

b. $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$ **[CBSE 2011]**

c. $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$ **[CBSE 2011]**

2. If a, b, c, are positive and unequal, show that the following determinant is negative:

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

3. $$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

4. $$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$ **[CBSE 2012]**

5. $$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

6. $$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$$

7. $$\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

8. If p, q, r are not in G.P and $\begin{vmatrix} 1 & \frac{q}{p} & \alpha+\frac{q}{p} \\ 1 & \frac{r}{q} & \alpha+\frac{r}{q} \\ p\alpha+q & q\alpha+r & 0 \end{vmatrix} = 0$, show that $p\alpha^2+2p\alpha+r=0$.

9. If a, b, c are real numbers, and $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

Show that either a + b +c = 0 or a = b = c.

QUESTIONS FOR SELF EVALUTION

1. Using properties of determinants, prove that : $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

2. Using properties of determinants, prove that : $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$

3. Using properties of determinants, prove that : $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

4. .Express $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

5. Let $A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$, prove by mathematical induction that : $A^n = \begin{bmatrix} 1-2n & -4n \\ n & 1+2n \end{bmatrix}$.

6. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find x and y such that $A^2 + xI = yA$. Hence find A^{-1}.

7. Let $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Prove that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

8. Solve the following system of equations : $x + 2y + z = 7$, $x + 3z = 11$, $2x - 3y = 1$.

9. Find the product AB, where $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.

10. Find the matrix P satisfying the matrix equation $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} P \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.