Class 12 Linear Differential Equation

Class 12th

Q.1)	Show that the general solution of the D.E. $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+x}=0$ is given by $x+y+1=A(x-x-y-2xy)$ where A is the parameter.
Sol.1)	We have, $\frac{dy}{dx}=-\frac{y^2+y+1}{x^2+x+1}$ $\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$ Interpreting both sides $\int\frac{dy}{(y^2+y+1)}=-\int\frac{dx}{(x^2+x+1)}$ $\Rightarrow \int\frac{1}{\left(y+\frac{1}{2}\right)^2-\frac{1}{4}+1}dy=-\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}+1}dx$ $\Rightarrow \int\frac{1}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dy=-\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$ $\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)=-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$ $\Rightarrow \frac{2}{\sqrt{3}}(\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)=c$ $\Rightarrow \tan^{-1}(\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1\left(\frac{2y+1}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}=\frac{\sqrt{3}}{2}c$ $\Rightarrow \tan^{-1}(\frac{\frac{2x+2y+2}{\sqrt{3}}}{\frac{3-4xy-2y-2x-1}{\sqrt{3}}}=\frac{\sqrt{3}}{2}c$ $\Rightarrow \frac{(2x+2y+2)\sqrt{3}}{2-4xy-2x-2y}=\tan\left(\frac{\sqrt{3}}{2}c\right)$ $\Rightarrow \frac{(x+y+1)}{1-2xy-x-y}=\frac{1}{\sqrt{3}}\tan\left(\frac{\sqrt{3}}{2}c\right)$ $\Rightarrow \frac{x+y+1}{1-2xy-x-y}=A;$ where $A=\frac{1}{\sqrt{3}}\tan\left(\frac{\sqrt{3}}{2}c\right)$ $\Rightarrow (x+y-1)=A(1-2xy-x-y)$ is the required solution ans.
Q.2)	Find the particular solution of the D.E. $(1+e^{2x})dy+(1+y^2)e^x dx=0$; given $x=0, y=1$
Sol.2)	We have, $(1+e^{2x})dy=-(1+y^2)e^x dx$ $\Rightarrow \frac{dy}{dx}=-\frac{1+y^2e^x}{1+e^x dx}$ Separating the variables & interpreting both sides $\Rightarrow \int\frac{dy}{1+y^2}=-\int\frac{ex}{1+e^{2x}}dx$ Put $e^x=t \Rightarrow e^x dx=dt$ $\Rightarrow \tan^{-1}y=-\int\frac{dt}{1+t^2}$ $\Rightarrow \tan^{-1}y=-\tan^{-1}t+c$ $\Rightarrow \tan^{-1}(y)+\tan^{-1}(e^x)=c$

	Put $x = 0$ & $y = 1$ $\Rightarrow \frac{r}{4} + \tan^{-1}(1) = c \Rightarrow c = \frac{r}{4} + \frac{r}{4} = \frac{r}{2}$ $\therefore \tan^{-1}(y) + \tan^{-1}(e^x) = \frac{r}{2}$ $\Rightarrow \tan^{-1}\left(\frac{y+e^x}{1-ye^x}\right) = \frac{r}{2}$ $\Rightarrow \frac{y+e^x}{1-ye^x} = \tan\left(\frac{r}{2}\right)$ $\Rightarrow \frac{y+e^x}{1-ye^x} = \frac{1}{0}$ .... $\left\{\tan\left(\frac{r}{2}\right) = \propto\right\}$ $\Rightarrow 0 = 1 - ye^x$ $\Rightarrow y = \frac{1}{e^x}$ is the required solution ans.
Q.3)	At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2,1)$.
Sol.3)	It is given that xy is the point of contact of the curve and its tangent. Hence the slope $= y + 3 + 4$. Let this be m_1 but we know that slope of the tangent to the curve is $\frac{dy}{dx}$ Let this be m_2 According to the given information, $m_2 = 2m_1$ $\frac{dy}{dx} = \frac{2(y+3)}{x+4}$ Seperating the variables we get $\frac{dy}{(y+3)} = \frac{2dx}{(x+4)}$ Integrating on both sides we get $\int \frac{dy}{(y+3)} = 2\int \frac{2dx}{(x+4)}$ or $\log(y+3) = 2\log(x+4) + \log c$ $\log(y+3) = \log c\,.(x+4)^2$ $y + 3 = c(x+4)^2$ It is given that the curve passes through the point $-2,1$ Substituting for xx and yy in the general equation to evaluate for cc we get $1 + 3 = c(-2+4)^2$ $4 = 4c$ $c = 1$ substituting this for c, we get $y + 3 = (x+4)^2$ is the required equation of the curve ans.
Q.4)	Solve the D.E. $\sqrt{1 + x^2 + y^2 + x^2y^2} = xy\frac{dy}{dx} = 0$
Sol.4)	We have, $\frac{dy}{dx} = -\frac{\sqrt{1+x^2+y^2+x^2y^2}}{xy}$

	$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{(1+x^2)(1+y^2)}}{xy}$ $\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1+x^2}\sqrt{1+y^2}}{xy}$ Separate the variables & interpreting both sides $\Rightarrow \int \frac{y}{\sqrt{1+y^2}}\, dx = -\int \frac{\sqrt{1+x^2}}{x} dx$ Put $1 + y^2 = t$ put $1 + x^2 = z^2$ $ydy = \frac{dt}{2}$ $2xdx = 2xdz$ $dx = \frac{zdz}{x}$ $\therefore \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\int \frac{z}{x}.\frac{zdx}{x}$ $\Rightarrow \frac{1}{2} \times 2\sqrt{t} = -\int \frac{z^2}{z^2-1} dz$ $\sqrt{t} = -\int \frac{z^2-1+1}{z^2-1} dz$ $\Rightarrow \sqrt{t} = -\int 1 + \frac{1}{z^2-1} dz$ $\Rightarrow \sqrt{1+y^2} = -\left[z + \frac{1}{2}\log\left\|\frac{z-1}{z+1}\right\|\right] + c$ $\Rightarrow \sqrt{1+y^2} = -\left[\sqrt{1+x^2} + \frac{1}{2}\log\left\|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right\|\right] + c$ $\Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} + \frac{1}{2}\log\left\|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right\| = c$ ans.
Q.5)	Find the equation of the curve passing through the point (1,0) given that slope of the tangent to the curve at any point (x, y) is $\frac{2x(\log x+1)}{\sin y+y\ \cos y}$.
Sol.5)	Slope of tangent at any point (x, y) is given by $\frac{dy}{dx}$ We have, $\frac{dy}{dx} = \frac{2x(\log x+1)}{\sin y+y\ \cos y}$ $\Rightarrow (\sin y + y\cos y)dy = 2x(\log x + 1)dx$ Interpreting both sides $\int(\sin y + y\cos y)dy = 2\int x(\log x + 1)dx$ $\Rightarrow \int \sin y\, dy + \int y\cos y\, dy = 2\int x\, dx + 2\int x\log x\, dx$ $\Rightarrow -\cos y + y\sin x - \int \sin y\, dy = \frac{2x^2}{2} + 2\left[\log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx\right]$ $\Rightarrow -\cos y + y\sin y + \cos y = y + 2\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + c$ $\Rightarrow y\sin y = x^2 + x^2\log x - \frac{x^2}{2} + c$ This equation passes through the point $(1,0)$ Put $x = 1$ and $y = 0$ $\Rightarrow 0 = 1 + 0 = -\frac{1}{2} + c$ $\Rightarrow c = -\frac{1}{2}$

	$\therefore y\sin y = x^2 + x^2\log x - \frac{x^2}{2} - \frac{1}{2}$ $\Rightarrow y\sin y = \frac{x^2}{2} + x62\log x - \frac{1}{2}$ $\Rightarrow 2y\sin y = x^2 + 2x^2\log x - 1$ is the required equation of curve. ans.
Q.6)	Solve the D.E. $3e^x\tan y\,dy + (1-e^x)\sec^2 y\,dy = 0$
Sol.6)	We have, $(1-e^x)\sec^2 y\,dy = -3e^x\tan y\,dx$ $\Rightarrow \frac{dy}{dx} = \frac{-3e^x\tan y}{(1-e^x)\sec^2 y}$ Separating variables & interpreting both sides $\Rightarrow \int\frac{\sec^2 y}{\tan y}dy = -3\int\frac{e^x}{1-e^x}dx$ Put $\tan y = t$ Put $1-e^x = z$ $\sec^2 ydy = dt$ $-e^x dx = dz$ $\Rightarrow \log\|t\| = 3\log z + \log c$ $\Rightarrow \log\left\|\frac{t}{z^3}\right\| = \log c$ Replace t & z $\Rightarrow \left\|\frac{\tan y}{(e^x-1)^3}\right\| = c$ $\Rightarrow \frac{\tan y}{(e^x-1)^3} = \pm c$ $\Rightarrow \tan y = c_1(e^x-1)^3$ where $c_1 = \pm c$ is the required solution. ans.
Q.7)	For the D.E. $xy\frac{dy}{dx} = (x+2)(y+2)$. Find the solution curve passes through (1,-1).
Sol.7)	We have, $\frac{dy}{dx} = \frac{(x+2)(y+2)}{xy}$ $\Rightarrow \frac{y}{y+2}dy = \frac{x+2}{x}dx$ Interpreting both sides $\Rightarrow \int\frac{y+2-2}{y+2}dy = \int\frac{x}{x}+\frac{2}{x}dx$ $\Rightarrow \int 1-\frac{2}{y+2}dy = \int 1+\frac{2}{x}dx$ $\Rightarrow y - 2\log\|y+2\| = x + 2\log\|x\| + c$ It is passes through the point (1,-1) Put $x = 1$ & $y = -1$ $\therefore -1 + 2\log\|1\| = 1 + 2\log\|1\| + c$ $\Rightarrow -1 + 0 = 1 + 0 = c \Rightarrow c = -2$ $\therefore y - 2\log\|y+2\| = x + 2\log\|x-2\|$ $\int\frac{y}{y+2}dy = \int\frac{x+2}{x}dx$ $\Rightarrow y = x + 2 = 2\log\|x\| + 2\log\|y+2\|$ $\Rightarrow y - x + 2 = \log\|x^2(y+2)^2\|$ $\Rightarrow y - x + 2 = \log(x^2(y+2)^2)$ ans.
Q.8)	In a bank principal increases at the rate of 5% per year. In how many years Rs.1000

	double itself.
Sol.8)	Let P be the principal at any time t Then, according to question $\frac{dp}{dt} = 5\frac{p}{100}$ $\Rightarrow \frac{dp}{dt} = \frac{p}{20}$ $\Rightarrow \frac{1}{p}dp = \frac{1}{20}dt$ Interpreting both sides $\int \frac{1}{p}dp = \frac{1}{20}\int dt$ $\Rightarrow \log p = \frac{1}{20}t + c$ $\Rightarrow p = e^{\frac{1}{20}t+c}$ $\Rightarrow p = e^{\frac{t}{20}}.e^{c}$ $p = e^{\frac{t}{20}} - c_1$ when $c_1 = e^c$ Given at $t = 0; p = 1000$ $\therefore 1000 = e^0.c_1$ $\Rightarrow c_1 = 1000$ $\therefore p = e^{\frac{t}{20}}.1000$ Let at $t = t\,; p = 2000$ $\Rightarrow 2000 = e^{\frac{t_1}{20}}.1000$ $\Rightarrow e^{\frac{t_1}{20}} = 2$ Taking log on both sides $\frac{t_1}{20} = \log 2$ $t_1 = 20\log_e 2\ years$ $\therefore$ principal doubles in $20\log_e 2\ years$ ans.
Q.9)	The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units & after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.
Sol.9)	Let the rate of change of the volume of balloon be k Hence $\frac{dv}{dt} = k$ $\frac{d}{dt}\left(\frac{4}{3}rr^3\right) = k$ $\frac{4}{3}r(3\pi^2).\left(\frac{dr}{dt}\right) = k$ Now seperating the variables, we get $4rr^2.dr = k.dt$ Integrating on both sides we get $4r\int r^2dr = k\int dt$

	$4r\left(\frac{r^3}{3}\right) = kt + C$ $4rr^3 = 3.(kt + C)$ Given $t = 0, r = 3t = 0, r = 3$ $4r(3^3) = 3(k.0 + C)$ $108r = 3C$ $C = 36r$ when $t = 3, r = 6t = 3, r = 6$ $4r.6^3 = 3.(k.3 + C)$ $864r = 3.(3k + 36r)$ Now Dividing throughout by 3 we get $3k = -288r - 36r$ $3k = 252r$ Hence $k = 84r$ Now substituting the values of k and c we get $4rr^3 = 3(84r.t + 36r)4r$ Taking 4r4r as a common factor $4r.r^3 = 4r.63t + 27$ Dividing throughout by $4r$ we get $r^3 = 63t + 27$ $r = (63t + 27)^{\frac{1}{3}}$ Thus the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$ ans.
Q.10)	In a culture the bacteria count is 1,00,000. The number is increases by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?
Sol.10)	From the given information we know that $\frac{dy}{dt}$ is proportional to y $\frac{dy}{dt} = ky$ on separating the variables we get $\frac{dy}{y} = k.dt$ Integrating on both sides we get $\int \frac{dy}{y} = k\int dt$ $\log y = kt + c$ .... (1) Let y_0 be the number of bacteria when $t = 0$ Hence $\log y_0 = C$ Substituting this value in equation (1) we get $\log y = kt + \log y_0$ $\log y - \log y_0 = kt$ $\log\left(\frac{y}{y_0}\right) = kt$ .... (2)

	It is also given that the number of bacteria increases 10% in 2 hours Hence $y = 110\frac{y_0}{100}$ Let y_0 be the number of bacteria when $t = 0$ $y_0 = \frac{11}{10}$ Substituting this in (2) we get $2k = log\left(\frac{11}{10}\right)$ or $k = \frac{1}{2}\log\frac{11}{10}$ Therefore $\frac{1}{2}\log\left(\frac{11}{10}\right)t = \log\left(\frac{y}{y_0}\right)$ $t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)}$ Let the time when the number of bacteria increases from 1,00,000 to 2,00,000 be t_1 $y = 2.y_0$ at $t = t_1$ Hence $t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ Hence in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ the number of bacteria increases from 1,00,000 to 2,00,000 ans.