	Class 12 Linear Differential Equation **Class 12^{th}**
Q.1)	Solve the D.E. $ydx + x\log y\, dy - x\log x\, dy - 2x\, dy = 0$
Sol.1)	We have, $y\, dx + x\, dy(\log y - \log x) - 2xdy = 0$ $\Rightarrow y\, dx + x\, dy. \log\left(\frac{y}{x}\right) - 2xdy = 0$ $\Rightarrow dy(x\log\frac{y}{x} - 2x) = -ydx$ $\Rightarrow \frac{dy}{dx} = -\frac{y}{x\log\left(\frac{y}{x}\right)-2x}$ …….(i) It is a homogeneous D.E. Put $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{xdv}{dx}$ put in eq. (i) $v + \frac{xdv}{dx} = \frac{-vx}{xlog\, v-2x}$ $\Rightarrow v + x\frac{dv}{dx} = \frac{-v}{\log v-2}$ $\Rightarrow x\frac{dv}{dx} = \frac{-v}{\log v-2} - v$ $\Rightarrow \frac{xdv}{dx} = \frac{-v-v\log v+2v}{\log v-2}$ $\Rightarrow \frac{xdv}{dx} = \frac{v-v\log v}{\log v-2}$ $\Rightarrow \frac{xdv}{dx} = \frac{-v(\log v-1)}{\log v-2}$ $\Rightarrow \frac{\log v-2}{v(\log v-1)} = -\frac{dx}{x}$ Interpreting both sides $\int\frac{\log v-2}{v(\log v-1)}\, dv = -\int\frac{dx}{x}$ Put $\log v - 1 = t$ $\frac{1}{v}dv = dt$ $\therefore \int\frac{t-1}{t}dt = -\log\lvert x\rvert$ $= \int 1 - \frac{1}{t}dt = -\log\lvert x\rvert$ $\Rightarrow t - \log t = -\log x + c$ $\Rightarrow \log v - 1 - \log v - 1 = -\log x + c$ $\Rightarrow \log\frac{vx}{\log v-1} = c + 1$ Replace v by $\frac{y}{x}$ $\Rightarrow \log\frac{y}{\log\left(\frac{y}{x}\right)-1} = c + 1$ $\Rightarrow \log\frac{y}{\log v-1} = c + 1$

	$\Rightarrow \frac{y}{\log\left(\frac{y}{x}\right)-1} = e^{c+1}$ $\Rightarrow \frac{y}{\log\left(\frac{y}{x}\right)-1} \pm e^{c+1}$ $\Rightarrow y = c_1\left(\log\left(\frac{y}{x}\right)-1\right)$; where $c_1 = \pm e^{c+1}$ is the required general solution ans.
Q.2)	Solve the D.E. $\left(xe^{\frac{y}{x}}+y\right)dx = xdy; y(1) = 1$
Sol.2)	We have, $\frac{dy}{dx} = \frac{xe^{\frac{y}{x}}+y}{x}$ ..... (i) Note: it is not homogeneous function D.E. Put $y = vx$ Diff. w.r.t. $x, \frac{dy}{dx} = v + \frac{xdv}{dx}$ put is eq.(i) $v + \frac{xdv}{dx} = \frac{xe^v+vx}{x}$ $\Rightarrow v + \frac{xdv}{dx} = e^v + v$ $\Rightarrow \frac{xdv}{dx} = e^v$ $\Rightarrow e^{-v}dv = \frac{dx}{x}$ $\Rightarrow \int -e^{-v} = \log\lvert x\rvert + c$ $\Rightarrow e^{-v} + \log\lvert x\rvert = -c$ Replace v by $\frac{y}{x}$ $e^{-\frac{y}{x}} + \log\lvert x\rvert = -c$ Put $x = 1$ and $y = 1$ $\Rightarrow e^{-1} + \log\lvert 1\rvert = -c$ $\Rightarrow \frac{1}{e} = -c$ put in above equation $\Rightarrow e^{-\frac{y}{x}} + \log\lvert x\rvert = \frac{1}{e}$ is the required solution ans.
Q.3)	Solve the D.E. $xy\log\left(\frac{x}{y}\right)dx + \left(y^2 - x^2\log\left(\frac{x}{4}\right)\right)dy = 0$
Sol.3)	We have, $xy\log\left(\frac{x}{y}\right)dx = -(y^2 - x^2\log\left(\frac{x}{y}\right)dy$ $\Rightarrow \frac{dx}{dy} = \frac{-\left(y^2-x^2\log\left(\frac{x}{y}\right)\right)}{xy\log\left(\frac{x}{y}\right)}$ It is a homogeneous D.E> Put $x = vy$ Diff. w.r.t $y, \frac{dx}{dy} = v + \frac{ydv}{dy}$ $\Rightarrow v + \frac{ydv}{dy} = \frac{-(y^2-v^2y^2\log v)}{vy^2\log v}$ $\Rightarrow v + \frac{ydv}{dy} = \frac{-(1-v^2\log v)}{v\log v}$

	$\Rightarrow \frac{ydv}{dy} = \frac{-(1-v^2\log v)}{v\log v} - v$ $\Rightarrow \frac{ydv}{dy} = \frac{-1+v^2\log v - v^2\log v}{v\log v}$ $\Rightarrow \frac{ydv}{dy} = \frac{-1}{v\log v}$ $\Rightarrow v\log v\ dv = \frac{-dy}{y}$ $\Rightarrow \int v\log v\,dv = -\int \frac{dy}{y}$ $\Rightarrow \log v.\frac{v^2}{2} - \int \frac{1}{v}.\frac{v^2}{2}dv = -\log\|y\|$ $\Rightarrow \frac{v^2}{2}\log v. - \frac{1}{2}\int v\,dv = -\log\|y\|$ $\Rightarrow \frac{v^2}{2}\log v - \frac{v^2}{4} = -\log\|y\| + c$ $\Rightarrow v^2\log v - \frac{v^2}{2} = -2\log\|y\| + 2c$ Replace v by $\frac{x}{y}$ $\Rightarrow \frac{x^2}{y^2}\log\left(\frac{x}{y}\right) - \frac{1}{2}\left(\frac{x^2}{y^2}\right) - 2\log\|y\| + 2c$ $\Rightarrow \frac{x^2}{y^2}\left(\log\left(\frac{x}{y}\right) - \frac{1}{2}\right) + \log\|y^2\| = 2c$ ans.
Reducible To Variable Separate Form (Put bracket = v)	
Q.4)	Solve the D.E. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$
Sol.4)	We have, $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ $\Rightarrow \frac{dy}{dx} = \sin(x+y)$ Put $x + y = v$ Diff. w.r.t x, $1 + \frac{dy}{dx} = \frac{dv}{dx}$ $\frac{dy}{dx} = \frac{dv}{dx} - 1$ $\therefore \frac{dv}{dx} - 1 = \sin v$ $\Rightarrow \frac{dv}{dx} = 1 + \sin v$ $\Rightarrow \int \frac{dv}{1+\sin v} = \int dx$ $\int \frac{1}{1+\sin x} \times \frac{1-\sin v}{1-\sin v}dv = \int dx$ (Rationalize) $\Rightarrow \int \frac{1-\sin v}{\cos^2 v}dv = x$ $\Rightarrow \int \sec^2 v - \tan v.\sec v.dv = x$ $\Rightarrow \tan v - \sec v = x + c$ Replace v by $x + y$ $\therefore \tan(x+y) - \sec(x+y) = x + c$ is the required solution ans.
Q.5)	Solve the D.E.

	$(x+y)^2\frac{dy}{dx}=a^2$
Sol.5)	We have, $\frac{dy}{dx}=\frac{a^2}{(x+y)^2}$ .... (i) Put $x+y=v$ Diff. w.r.t x, $1+\frac{dy}{dx}=\frac{dv}{dx}$ $\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$ $\therefore$ equation (i) becomes $\frac{dv}{dx}-1=\frac{a^2}{v^2}$ $\Rightarrow \frac{dv}{dx}=\frac{a^2}{v^2}+1$ $\Rightarrow \frac{dv}{dx}=\frac{a^2+v^2}{v^2}$ $\Rightarrow \frac{v^2}{v^2+a^2}dv=dx$ Interpreting both sides $\int \frac{v^2}{v^2+a^2}dv=\int dx$ $\Rightarrow \int \frac{v^2+a^2-a^2}{v^2+a^2}dv=\int dx$ $\Rightarrow \int 1-\frac{a^2}{v^2+a^2}dv=\int dx$ $\Rightarrow v-a^2\times\frac{1}{a}\tan^{-1}\left(\frac{v}{a}\right)=x+c$ Replace v by $(x+y)$ $\Rightarrow x+y-a\tan^{-1}\left(\frac{x+y}{a}\right)x+c$ $\Rightarrow y=a\tan^{-1}\left(\frac{x+y}{a}\right)+c$ is the required solution ans.
Q.6)	Solve $\frac{dy}{dx}=\cos(x+y)+\sin(x+y)$
Sol.6)	Put $x+y=v$ Diff. w.r.t x, $1+\frac{dy}{dx}=\frac{dv}{dx}$ $\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$ $\therefore \frac{dv}{dx}-1=\cos v+\sin v$ $\Rightarrow \frac{dv}{dx}=1+\cos v+\sin v$ $\Rightarrow \int \frac{1}{1+\sin v+\cos v}dv=\int dx$ $(Type\ single\ \sin x,\cos x)$ $\Rightarrow \int \frac{1}{1+\frac{2\tan\frac{v}{2}}{1+\tan^2\frac{v}{2}}+\frac{1-\tan^2\frac{v}{2}}{1+\tan^2\frac{v}{2}}}dv=x$ $\Rightarrow \int \frac{1+\tan^2\frac{v}{2}}{1+\tan^2\left(\frac{v}{2}\right)+2\tan\left(\frac{v}{2}\right)+1-\tan^2\left(\frac{v}{2}\right)}dv=x$

	$\Rightarrow \int \frac{\sec^2\left(\frac{v}{2}\right)}{2(1+\tan\frac{v}{2})} dv = x$ Put $1 + \tan\frac{v}{2} = t$ $\sec^2\left(\frac{v}{2}\right).\frac{1}{2}dv = dt$ $\Rightarrow \int \frac{dt}{t} = x$ $\Rightarrow \log\left\|1 + \tan\frac{v}{2}\right\| = x + c$ $\Rightarrow \log\left\|1 + \tan\left(\frac{x+y}{2}\right)\right\| = x + c$ ans.
Q.7)	Solve the initial problem $(x-y)(dx+dy) = dx - dy\,; y(0) = -1$
Sol.7)	We have, $xdx + xdy - ydx - ydy = dx - dy$ $\Rightarrow dy(x-y+1) = dx(1-x+y)$ $\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$ $\Rightarrow \frac{dy}{dx} = \frac{1-(x-y)}{(x-y)+1}$ Put $x - y = v$ Diff. w.r.t. x, $1 - \frac{dy}{dx} = \frac{dv}{dx}$ $\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$ $\therefore 1 - \frac{dv}{dx} = \frac{1-v}{v+1}$ $\Rightarrow= \frac{v+1-1+v}{v+1}$ $\Rightarrow \frac{dv}{dx} = \frac{2v}{v+1}$ $\Rightarrow \frac{v+1}{v}\ dv = 2dx$ Interpreting both sides $\Rightarrow \int \frac{v+1}{v} dv = 2\int dx$ $\Rightarrow \int 1 + \frac{1}{v} dv = 2x$ $\Rightarrow v + \log\|v\| = 2x + c$ $\Rightarrow x - y + \log\|x-y\| = 2x + c$ $\Rightarrow \log\|x-y\| = x + y + c$ Put $x = 0$ & $y = -1$ $\Rightarrow \log\|1\| = 0 - 1 + c$ $\Rightarrow 0 = -1 + c \Rightarrow c = 1$ $\therefore \log\|x-y\| = x + y + 1$ $\Rightarrow \|x-y\| = e^{x+y+1}$ is the required solution ans.
	Variable Separate Form
Q.8)	Solve the D.E. $(x+1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0$

Sol.8)	We have, $\frac{dy}{dx} = \frac{2e^{-y}-1}{x+1}$ Separating the variables $\Rightarrow \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$ Interpreting both sides $\int \frac{dy}{2e^{-y}-1} = \int \frac{dx}{x+1}$ $\Rightarrow \int \frac{e^y}{2-e^y} = \int \frac{dx}{x+1}$ Put $2 - e^y = t$ $-e^y dy = dt$ $e^y dy = -dt$ $\Rightarrow -\log\|2 - e^y\| = \log\|x + 1\| + \log c$ $\Rightarrow \log\left\|\frac{1}{2-e^y}\right\| = \log\|c(x + 1)\|$ $\Rightarrow \left\|\frac{1}{2-e^y}\right\| = \|c(x + 1)\|$ Put $x = 0$ & $y = 0$ $\Rightarrow \left\|\frac{1}{2-1}\right\| = \|c(0 + 1)\| \Rightarrow c = 1$ $\Rightarrow \left\|\frac{1}{2-e^y}\right\| = \|x + 1\|$ $\Rightarrow \|(x + 1)(2 - e^y)\| = 1$ $\Rightarrow (x + 1)(2 - e^y) = \pm 1$ But $x = 0, y = 0$ does not satisfy the solution $(x + 1)(2 - e^y) = -1$ $\therefore (x + 1)(2 - e^y) = 1$ $\Rightarrow -\int \frac{dt}{t} = \log\|x + 1\|$ $\Rightarrow 2 - e^y = \frac{1}{x+1}$ $\Rightarrow e^y = \frac{2-1}{x+1}$ $\Rightarrow e^y = \frac{2x+1}{x+1}$ $\Rightarrow y = \log\left(\frac{2x+1}{x+1}\right)$ is the required solution ans.
Q.9)	Solve the D.E. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$
Sol.9)	We have, $y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$ $\Rightarrow \frac{dy}{dx}(a + x) = y - ay^2$ $\Rightarrow \frac{dy}{dx} = \frac{-ay^2-y}{a+x}$ Separating variable & interpreting both sides $\int \frac{dy}{ay^2-y} = -\int \frac{dx}{a+x}$ $\Rightarrow \frac{1}{a}\int \frac{1}{y^2-\frac{y}{a}} dy = -\log\|x + a\|$

	$\Rightarrow \frac{1}{a}\int \frac{1}{y-\left(\frac{1}{2a}\right)^2-\left(\frac{1}{2a}\right)^2} = -\log\lvert x+a\rvert$ $\Rightarrow \frac{1}{a} \times \frac{1}{2\times\frac{1}{2a}} \log\left\lvert\frac{y-\frac{1}{2a}-\frac{1}{2a}}{y-\frac{1}{2a}-\frac{1}{2a}}\right\rvert = -\log\lvert x+a\rvert + \log c$ $\Rightarrow \log\left\lvert\frac{ay-1}{ay}\right\rvert + \log\lvert x+a\rvert = \log c$ $\Rightarrow \log\left\lvert\frac{(ay-1)(x+a)}{ay}\right\rvert = \log c$ $\Rightarrow \left\lvert\frac{(ay-1)(x+a)}{ay}\right\rvert = c$ $\Rightarrow \frac{(ay-1)(x+a)}{ay} = \pm c$ $\Rightarrow (ay-1)(x+a) = \pm acy$ $\Rightarrow (ay-1)(x+a) = c_1 y$; where $c_1 = \pm ac$ is the required solution ans.
Q.10)	Solve the initial value problem $\log\left(\frac{dy}{dx}\right) = 3x + 4y$; $x = 0$ and $y = 0$.
Sol.10)	We have, $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ $\Rightarrow \frac{dy}{dx} = e^{3x+4y}$ $\Rightarrow \frac{dy}{dx} = e^{3x}.e^{4y}$ $\Rightarrow \int e^{-4y}\,dv = \int e^{3x}\,dx$ $\Rightarrow -\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + c$ Put $x = 0$ & $y = 0$ $\Rightarrow -\frac{1}{4} = \frac{1}{3} + c$ $c = -\frac{7}{12}$ $\therefore\ -\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} - \frac{7}{12}$ $\Rightarrow -\frac{1}{4}e^{-4y} = \frac{4e^{3x}-7}{12}$ $\Rightarrow -3xe^{-4y} = 4e^{3x} - 7$ $\Rightarrow 4e^{3x} = 3e^{-4y} = 7$ is the required solution ans.