Class 12 Linear Differential Equation

Class 12th

Q.1) Find the general solution of the D.E. $\frac{dy}{dx} + x\sin(2y) = x^3 . \cos^2 y$.

Sol.1) We have, $\frac{dy}{dx} + x\sin(2y) = x^3\cos^2 y$

Divide by $\cos^2 y$

$\Rightarrow \sec^2 y . \frac{dy}{dx} + x\sin(2y) = x^3\cos^2 y$

$\Rightarrow \sec^2 y . \frac{dy}{dx} + x\sin\frac{2y}{\cos^2 y} = x^3$

$\Rightarrow \sec^2 y\frac{dy}{dx} + x . \frac{2\sin y\cos y}{\cos^2 y} = x^3$

$\Rightarrow \sec^2\frac{dy}{dx} + 2x\tan y = x^3$

Let $\tan y = v \;\Rightarrow \frac{\sec^2 y dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx} + 2xv = x^3$

This is a linear D.E. of the form $\frac{dv}{dx} + Pv = \theta$

Here $P = 2x$ and $\theta = x^3$

$I.F. = e^{\int 2xdx} = e^{x^2}$

Solution is given by $v.(I.F.) = \int\theta(I.F.)dx + C$

$\Rightarrow ve^{x^2} = \int\theta\,(I.F.)dx + C$

$\Rightarrow ve^{x^2} = \int x^3 . e^{x^2}dx + C$

$\Rightarrow ve^{x^2} = \int x^2 . e^{x^2} . xd + C$

Put $x^2 = t \Rightarrow xdx = \frac{dt}{2}$

$\Rightarrow ve^{x^2} = \frac{1}{2}\int t.e^t dt + C$

$\Rightarrow ve^{x^2} = \frac{1}{2}[t - e^t - \int e^t dt] + C$

$\Rightarrow ve^{x^2} = \frac{1}{2}(te^t - e^t) + C$

$\Rightarrow ve^{x^2} = \frac{e^t}{2}(t-1) + C$

Re-pairing v and t by $\tan y$ and x^2 respectively.

$\Rightarrow \tan y . e^{x^2} - \frac{1}{2}e^{x^2}(x^2 - 1) + C$ ans.

Q.2) Solve the D.E. $(x^2 - 1)\frac{dy}{dx} + 2(x+2)y = 2(x+1)$

Sol.2) Divide by $(x^2 - 1)$

$\frac{dy}{dx} + \frac{2(x+2)}{x^2-1}y = \frac{2(x+1)}{x^2-1}$

Compare with $\frac{dy}{dx} + Py = \theta$

We have, $P = \frac{2(x+2)}{x^2-1}$ and $\theta = \frac{2}{x-1}$

	$I.F. = e^{\int P dx} = e^{\int \frac{2(x+2)}{x^2-1}dx}$ Let $I = \int \frac{2x+4}{x^2-1}dx$ $= \int \frac{2x}{x^2-1}dx + 4\int \frac{1}{x^2-1}dx$ Put $x^2 - 1 = t; 2xdx = dt$ $\Rightarrow I = \int \frac{dt}{t} + 4 \times \frac{1}{2}\log\left\|\frac{x-1}{x+1}\right\|$ $I = \log\|x^2-1\| + 2\log\left\|\frac{x-1}{x+1}\right\|$ $= \log\|x^2-1\| + \log\frac{(x-1)^2}{(x+1)^2}$ $= \log\left(\frac{(x-1)^2}{(x+1)^2}(x+1)(x-1)\right)$ $I = \log\left(\frac{(x-1)^3}{x+1}\right)$ $\therefore I.F. = e^{\log\left(\frac{(x-1)^3}{x+1}\right)}$ $I.F. = \frac{(x-1)^3}{x+1}$ New solution is given by $y.(I.F.) = \int \theta\,(I.F.)dx + C$ $\Rightarrow y\frac{(x-1)^3}{x+1} = \int \frac{2}{(x-1)}.\frac{(x-1)^3}{(x+1)}dx + C$ $\Rightarrow y\frac{(x-1)^3}{x+1} = 2\int \frac{(x-1)^2}{x+1}dx + C$ $\Rightarrow y\frac{(x-1)^3}{x+1} = 2\int \frac{x^2-2x+1}{x+1}dx + C$ $\Rightarrow y\frac{(x-1)^3}{x+1} = 2\int (x-3) + \frac{4}{x+1}dx + C$ $\Rightarrow y\frac{(x-1)^3}{x+1} = 2\left(\frac{x^2}{2} - 3x + 4\log\|x+1\|\right) + C$ $\Rightarrow y = \frac{2(x+1)}{(x-1)^3}\left(\frac{x^2}{2} - 3x + 4\log\|x+1\|\right) + C$ ans.
Q.3)	Find the particular solution $\frac{dy}{dx} - 3y\cot x = \sin(2x)\,; y = 2$ when $x = \frac{r}{2}$
Sol.3)	Compare with $\frac{dy}{dx} + Py = \theta$ We have, $P = -3\cot x$ and $\theta = \sin(2x)$ $I.F. = e^{-3\int \cot x\,dx} = e^{-3\log/\sin x}$ $\Rightarrow I.F. = e^{\log(\sin x)^{-3}} = \frac{1}{\sin^3 x} \Rightarrow IF = \frac{1}{\sin^3 x}$ Solution given by $y = (I.F.) = \int \theta\,(IF)dx + C$ $\Rightarrow y\frac{31}{\sin^3 x} = \int \sin(2x).\frac{1}{\sin^3 x}dx + C$ $\Rightarrow \frac{y}{\sin^3 x} = 2\int \sin x.\cos x.\frac{1}{\sin^3 x}dx + C$ $\Rightarrow \frac{y}{\sin^3 x} 2\int \cot x.cosec\ x\ dx + C$ $\Rightarrow \frac{y}{\sin^3 x} = 2\,(-cosec\ x) + C$

	$\Rightarrow y = -2\,cosec\,x.\sin^3 x + C\sin^3 x$ $\Rightarrow y = -2\sin^2 x + C.\sin^3 x$ Put initial condition $x = \frac{r}{2}$ and $y = 2$ $\Rightarrow 2 = -2\sin^3\left(\frac{r}{2}\right) + C.\sin^3\left(\frac{r}{2}\right)$ $\Rightarrow 2 = -2 + C \Rightarrow C = 4$ $\therefore$ particular solution is given by $y = -2\sin^2 x + 4\sin^3 x$ ans.
Q.4)	Solve the D.E. $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x\,; y(0) = 1$
Sol.4)	Compare with $\frac{dy}{dx} + Py = \theta$ We have $P = \tan x\,, \theta = 2x + x^2\tan x$ $I.F. = e^{\int Pdx} = e^{\int \tan x\,dx} = e^{\log(\sec x)} = \sec x$ Solution is given by $y(\sec x) = \int (x^2\tan x + 2x)\sec x\,dx + C$ $\Rightarrow y\sec x = \int x^2\tan x\sec x\,dx + \int 2x\sec x\,dx + C$ $\Rightarrow y\sec x = x^2.\sec x - 2\int x.\sec x + 2\int x\sec x\,dx + C$ $\Rightarrow y\sec x = x^2\sec x + C$ Put $y = 1$ and $x = 0$ $\Rightarrow 1.\sec(0) = 0 + C$ $\Rightarrow 1 = C$ $\therefore$ solution is given by $y\sec x = x^2\sec x + 1$ Or $y = x^2 + \cos x$ ans.
Q.5)	Find one-parameter families of solution curve of the D.E. (or solve the D.E.) $\sec x\frac{dy}{dx} + y = e^{\sin x}$
Sol.5)	Divide by $\sec x$ $\frac{dy}{dx} + y\cos x = e^{\sin x}.\cos x$ Here $P = \cos x\,; \theta = e^{\sin x}.\cos x$ $I.F. = e^{\int Pdx} = e^{\int \cos x dx} = e^{\sin x}$ Solution is given by $y(I.F.) = \int \theta\,(I.F.)dx + C$ $\Rightarrow y\,e^{\sin x} = \int e^{\sin x}.\cos x.e^{\sin x}dx + C$ $\Rightarrow y\,e^{\sin x} = \int e^{2\sin x}.\cos x\;dx + C$ Put $\sin x = t \Rightarrow \cos x\,dx = dt$ $\Rightarrow\; y\,e^{\sin x} = \int e^{2t}\,dt + C$ $\Rightarrow ye^{\sin x} = \frac{1}{2}e^{2t} + C$ $\Rightarrow ye^{\sin x} = \frac{1}{2}e^{2\sin x} + C$ ans.
Q.6)	Solve the D.E. $y\,dx - (x + 2y^2)dy = 0$
Sol.6)	$y\,dx = (x + 2y^2)dy$ $y\frac{dx}{dy} = x + 2y^2$

	$\Rightarrow \frac{dx}{dy} = \frac{x}{y} = 2y$ Comparing with $\frac{dx}{dy} + Px = \theta$ We have, $P = -\frac{1}{y}; \theta = 2y$ $I.F. = e^{\int P dy} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$ $\therefore I.F. = \frac{1}{y}$ Solution is given by $x.(I.F.) = \int \theta.(I.F.)dy + c$ $\Rightarrow x.\frac{1}{y} = \int 2y\left(\frac{1}{y}\right)dy + c$ $\Rightarrow \frac{x}{y} = 2y + c$ $\Rightarrow x = 2y^2 + cy$ is the required solution ans.
Q.7)	Solve $\frac{\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)dx}{dy} = 1$.
Sol.7)	We have $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$ $\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ Comparing with $\frac{dy}{dx} + Py = \theta$ Here $P = \frac{1}{\sqrt{x}}$ and $\theta = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $I.F. = e^{\int \frac{1}{\sqrt{x}}}$ and $\theta = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $I.F. = e^{\int \frac{1}{\sqrt{x}} dx}$ and $e^{2\sqrt{x}} \Rightarrow I.F. = e^{2\sqrt{x}}$ Solution is given by $y.(I.F.) = \int \theta\,(I.F.)dx + c$ $\Rightarrow y\,.e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}} dx + c$ $\Rightarrow y.e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + c$ $\Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + c$ ans.
Q.8)	Solve the initial value problem $(1 + y^2)dx = (\tan^{-1} y - x)dy; y(0) = 0$
Sol.8)	We have $(1 + y^2)dx = (\tan^{-1} y - x)dy$ $\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}$ $\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$ $\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$ Comparing with $\frac{dx}{dy} + Px = \theta$

	Here $P = \frac{1}{1+y^2}$ and $\theta = \frac{\tan^{-1} y}{1+y^2}$ $I.F. = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy}$ $I.F. = e^\wedge \tan^{-1} y$ Solution is given by $x.(I.F.) = \int \theta(I.F.)dy + c$ $\Rightarrow x.e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c$ Put $\tan^{-1} y = t \Rightarrow \frac{1}{1+y^2} dy = dt$ $\therefore xe^{\tan^{-1} y} = t\, e^t\, dt + c$ $\Rightarrow x\, e^{\tan^{-1} y} = t\, e^t - \int 1.e^t dt + c$ $\Rightarrow xe^{\tan^{-1} y} = t\, e^t - e^t + c$ $\Rightarrow xe^{\tan^{-1} y} = e^\wedge \tan^{-1} y\,(\tan^{-1} y - 1) + c$ Put initial condition $x = 0$ and $y = 0$ $\Rightarrow 0 = e^0(0 - 1) + c$ $\Rightarrow 0 = -1 + c \Rightarrow c = 1$ $\therefore xe^{\tan^{-1} y} = e^{\tan^{-1} y(\tan^{-1} y - 1)} + 1$ $\Rightarrow x = (\tan^{-1} y - 1) + e^{-\tan^{-1}} y$ ans.
Q.9)	Find the particular solution of the DE $\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y\,; y(0) = \frac{r}{2}$
Sol.9)	Comparing with $\frac{dx}{dy} + Px = \theta$ We have $P = \cot y\,;\ \theta = y^2 \cot y$ $I.F. = e^{\int \cot y\, dy} = e^{\log(\sin y)} = \sin y$ Solution is given by $x.(I.F.) = \int \theta\,(IF)dy + c$ $\Rightarrow x.\sin y = \int (2y + y^2 \cot y) \sin y\ \ dy + c$ $\Rightarrow x.\sin y = \int 2y \sin y\, dy + \int y^2 . \cos y\ \ dy + c$ $\Rightarrow x.\sin y = 2\int y \sin y\, dy + y^2 \sin y - 2$ $\Rightarrow x.\sin y = y^2 \sin y + c$ $\Rightarrow 0 = \frac{r^2}{4}(1) + c$ $\Rightarrow c = -\frac{r^2}{4}$ $\therefore x \sin y = y^2 \sin y - \frac{r^2}{4}$ is the required solution ans.
Q.10)	Solve the DE $ye^y dx = (y^3 + 2xe^y)dy; y(0) = 1.$
Sol.10)	Divide by dy $\frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y}$ $\Rightarrow \frac{dx}{dy} = \frac{y^2}{e^y} + \frac{2x}{y}$

	$\Rightarrow \frac{dx}{dy} - \frac{2x}{y} = y^2 e^{-y}$ Comparing with $\frac{dx}{dy} + Py = \theta$ We have $P = -\frac{2}{y}; \theta = y^2 e^{-y}$ $I.F. = e^{-\int \frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$ $I.F. = \frac{1}{y^2}$ Solution is given by $x.(I.F.) = \int \theta\, (I.F.) dy + c$ $\Rightarrow x.\frac{1}{y^2} = \int y^2.e^{-y}.\frac{1}{y^2} dy + c$ $\Rightarrow \frac{x}{y^2} = \frac{e^{-}}{-1} + c$ $\Rightarrow \frac{x}{y^2} = -\frac{1}{e^y} + c$ Put $x = 0$ & $y = 1$ $\Rightarrow \therefore \frac{x}{y^2} = -\frac{1}{e^y} + \frac{1}{e}$ $\Rightarrow x = y^2(e^{-1} - e^{-y})$ is the required solution ans.