CHAPTER – 7

INTEGRALS

I. Integrals of Type : $\int \sqrt{ax^2 + bx + c}\ dx$

In order to evaluate the above type of integral, we put $ax^2 + bx + c$ in the form :

$$\begin{cases} a\left[\left(x + \dfrac{b}{2a}\right)^2 + \left(\dfrac{\sqrt{4ac - b^2}}{2a}\right)^2\right] & \text{when } b^2 < 4ac. \\ a\left[\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] & \text{when } b^2 > 4ac. \end{cases}$$

and the integral reduces to one of the following three forms:

$\int \sqrt{a^2 + x^2}dx$, $\int \sqrt{a^2 - x^2}dx$, $\int \sqrt{x^2 - a^2}dx$, which can be evaluated by using standard formulae.

Example 1: Evaluate

(i) $\int \sqrt{x^2 + 4x + 8}\ dx$, (ii) $\int \sqrt{(x - 5)(7 - x)}\ dx$

(ii) $\int \sqrt{14x - 20 - 2x^2}\ dx$ (iv) $\int \sqrt{4a - x^2}\ dx$

Solution : (i) $\int \sqrt{x^2 + 4x + 8}\ dx = \int \sqrt{x^2 + 4x + 4 + 4}\ dx$

$$= \int \sqrt{(x+2)^2 + (2)^2}\ dx$$

$$= \frac{x+2}{2}\sqrt{x^2 + 4x + 8} + \frac{4}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

$$= \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 8} + 2\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

(ii) $\int \sqrt{(x - 5)(7 - x)}\ dx = \int \sqrt{12x - 35 - x^2}\ dx$

$$= \int \sqrt{-35 - (x^2 - 12x)}dx = \int \sqrt{-35 - (x^2 - 12x + 36 - 36)}\ dx$$

$$= \int \sqrt{(36 - 35) - (x - 6)^2}dx = \int \sqrt{1^2 - (x - 6)^2}\ dx$$

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$$= \frac{x-6}{2}\sqrt{(x-5)(7-x)} + \frac{1}{2}\sin^{-1}\left(\frac{x-6}{1}\right) + c$$

(iii) $\int \sqrt{14x - 20 - 2x^2}\, dx = \sqrt{2}\int \sqrt{-10 + 7x - x^2}\, dx$

$$= \sqrt{2}\int \sqrt{-10 - (x^2 - 7x)}\, dx = \sqrt{2}\int \sqrt{-10 - \left(x^2 - 7x + \frac{49}{4} - \frac{49}{4}\right)}\, dx$$

$$= \sqrt{2}\int \sqrt{-10 + \frac{49}{4} - \left(x - \frac{7}{2}\right)^2}\, dx = \sqrt{2}\int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{x - 7/2}{2}\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - 7/2\right)^2} + \frac{9}{8}\sin^{-1}\left(\frac{x - \frac{7}{2}}{\frac{3}{2}}\right)\right] + c$$

$$= \sqrt{2}\left[\frac{2x-7}{4}\sqrt{7x - 10 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right)\right] + c$$

(iv) $\int \sqrt{4ax - x^2}\, dx = \int \sqrt{-\left(x^2 - 4ax + 4a^2 - 4a^2\right)}\, dx$

$$= \int \sqrt{4a^2 - (x^2 - 4ax + 4a^2)}\, dx = \int \sqrt{(2a)^2 - (x - 2a)^2}\, dx$$

$$= \frac{x-2a}{2}\sqrt{4ax - x^2} + \frac{4a^2}{2}\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

$$= \frac{1}{2}(x-2a)\sqrt{4ax - x^2} + 2a^2\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

II. Integrals of Type : $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

Here, px + q is written as

$px + q = A\left[\frac{d}{dx}\left(ax^2 + bx + c\right)\right] + B$ and the values of A and B are determined by equating the coefficients of x and constant terms on both sides.

Then, writing

$$\int (px+q)\sqrt{ax^2 + bx + c}\, dx = A\int (2ax + b)\sqrt{ax^2 + bx + c}\, dx + B\int \sqrt{ax^2 + bx + c}\, dx$$

$$= \frac{2A}{3}(ax^2 + bx + c)^{3/2} + B\int \sqrt{ax^2 + bx + c}\ dx$$

The second part is evaluated as explained in above example.

Example 2: Evaluate

(i) $\int (x-3)\sqrt{x^2+4x+3}\ dx$ (ii) $\int (3x+5)\sqrt{2x^2+3x+7}\ dx$

(ii) $\int (x-4)\sqrt{4+3x-x^2}\ dx$ (iii) $\int (5x-1)\sqrt{6+5x-2x^2}\ dx$

Solution :

(i) Let $I = \int (x-3)\sqrt{x^2+4x+3}\ dx$

$$x - 3 = A(2x+4) + B$$

$$\Rightarrow A = \frac{1}{2},\ B = -5$$

$$\therefore I = \int \frac{1}{2}(2x+4)\sqrt{x^2+4x+3}\ dx - 5\int \sqrt{x^2+4x+3}\ dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\int \sqrt{(x+2)^2 - (1)^2}\ dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\left[\frac{x+2}{2}\sqrt{x^2+4x+3}\ \frac{-1}{2}\log\left|(x+2) + \sqrt{x^2+4x+3}\right|\right] + c$$

(ii) $I = \int (3x+5)\sqrt{2x^2+3x+7}\ dx$

$$3x + 5 = A(4x+3) + B \Rightarrow 4A = 3 \text{ and } 3A + B = 5$$

$$\Rightarrow A = 3/4 \text{ and } B = \frac{11}{4}$$

$$I = \int \frac{3}{4}(4x+3)\sqrt{2x^2+3x+7}\ dx + \frac{11}{4}\int \sqrt{2}\sqrt{x^2 + \frac{3}{2}x + \frac{7}{2}}\ dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\int \sqrt{\left(x + 3/4\right)^2 + \frac{7}{2} - \frac{9}{16}}\ dx$$

$$= \quad \frac{1}{2}\left(2x^2+3x+7\right)^{3/2}+\frac{11\sqrt{2}}{4}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2}\,dx$$