

Integration (Indefinite Integrals)

| | |
|--------|---|
| Q.1) | $(a) I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4 \sin x} dx$ $(b) I = \int \frac{1}{2e^{2x} + 3e^x + 1} dx$ |
| Sol.1) | $(a) I = \int \frac{2\sin 2x - \cos x}{6 - \cos^2 x - 4 \sin x} dx$ $= \int \frac{4 \sin x \cdot \cos x - \cos x}{6 - \cos^2 x \cdot 4 \sin x} dx$ $= \int \frac{(4 \sin x - 1) \cos x}{6 - (1 - \sin^2 x) - 4 \sin x} dx$ <p>put $\sin x = t$</p> <p>$\cos x \ dx = dt$</p> $I = \int \frac{(4t - 1)}{6 - (1 - t^2) - 4t} dt$ $= \int \frac{4t-1}{t^2-4t+5} dt$ <p>Proceed Yourself</p> $2\log \sin^2 x - 4 \sin x + 5 + 7\tan^{-1}(\sin x - 2) + c \quad \text{ans.}$ $(b) I = \int \frac{1}{2e^{2x} + 3e^x + 1} dx$ $= \int \frac{1}{e^{-2x} + \frac{3}{e^{-x}} + 1} dx$ <p>L.C.M</p> $= \int \frac{e^{-2x}}{2+3e^{-x}+e^{-2x}} dx$ $= \int \frac{e^{-x} \cdot e^{-x}}{2+3e^{-x}+e^{-2x}} dx$ <p>put $e^{-x} = t$</p> <p>$e^{-x} dx = -dt$</p> $\therefore I = - \int \frac{t}{2+3t+t^2} dt$ $= -2 \int \frac{2t+3-3}{t^2+3t+2} dt \quad (\text{to make } 2t+3)$ $= -2 \int \frac{2t+3}{t^2+3t+2} dt + 6 \int \frac{1}{t^2+3t+2} dt$ <p>put $t^2 + 3t + 2 = z$</p> <p>$(2t+3)dt = dz$</p> $\therefore I = -2 \int \frac{dz}{z} + 6 \int \frac{1}{(t+\frac{3}{2})^2 - \frac{9}{4} + 2} dt$ $= -2\log z + 6 \int \frac{1}{(t+\frac{3}{2})^2 - (\frac{1}{2})^2} dt$ $= -2\log t^2 + 3t + 2 + 6 \times \frac{1}{2 \times \frac{1}{2}} \log \left \frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}} \right + c$ $= -2\log t^2 + 3t + 2 + 6\log \left \frac{2t+2}{2t+4} \right + c$ |



| | | |
|--------|--|---|
| | replacing t by e^{-x} = $I = -2\log e^{-2x} + 3e^{-x} + 2 + 6\log \left \frac{e^{-x}+1}{e^{-x}+2} \right + c$ ans. | |
| Q.2) | (a) $I = \int \frac{3x-1}{\sqrt{1-x-x^2}} dx$ | (b) $I = \int \sqrt{\frac{1+x}{x}} dx$ |
| Sol.2) | $ \begin{aligned} (a) I &= \int \frac{3x-1}{\sqrt{1-x-x^2}} dx \\ &= 3 \int \frac{x-\frac{1}{3}}{\sqrt{1-x-x^2}} dx \quad (\text{to make } (-2x-1)) \\ &= \frac{-3}{2} \int \frac{-2x+\frac{2}{3}}{\sqrt{1-x-x^2}} dx \\ &= \frac{-3}{2} \int \frac{-2x+\frac{2}{3}-1+1}{1-x-x^2} dx \\ &= \frac{-3}{2} \int \frac{(-2x-1)+\frac{5}{3}}{\sqrt{1-x-x^2}} dx \\ &= \frac{-3}{2} \int \frac{(-2x-1)}{\sqrt{1-x-x^2}} dx - \frac{5}{2} \int \frac{1}{\sqrt{1-x-x^2}} dx \end{aligned} $ <p>put $1-x-x^2 = t$ $(-2x-1)dx = dt$</p> $ \begin{aligned} \therefore I &= \frac{-3}{2} \int \frac{dt}{\sqrt{t}} - \frac{5}{2} \int \frac{1}{\sqrt{-[x^2+x-1]}} dx \\ &= \frac{-3}{2} \times 2\sqrt{t} - \frac{5}{2} \int \frac{1}{\sqrt{-[(x+\frac{1}{2})^2 - \frac{1}{4} - 1]}} dx \\ &= -3\sqrt{t} - \frac{5}{2} \int \frac{1}{\sqrt{-[(x+\frac{1}{2})^2 - (\frac{\sqrt{5}}{2})^2]}} dx \\ &= -3\sqrt{1-x-x^2} - \frac{5}{2} \int \frac{1}{\sqrt{(\frac{\sqrt{5}}{2})^2 - (x+\frac{1}{2})^2}} dx \\ &= -3\sqrt{1-x-x^2} - \frac{5}{2} \sin^{-1} \left(\frac{2x+1}{\sqrt{5}} \right) + c \quad \text{ans.} \end{aligned} $ | $ (b) I = \int \sqrt{\frac{1+x}{x}} dx \\ = \int \sqrt{\frac{1+x}{x} \times \frac{1+x}{1+x}} dx \\ = \int \frac{1+x}{\sqrt{x+x^2}} dx \quad (\text{make } 2x+1) $ <p>Proceed yourself $\sqrt{x^2+x} + \frac{1}{2} \log \left \left(x + \frac{1}{x}\right) + \sqrt{x^2+x} \right + c$ ans.</p> |
| Q.3) | (a) $I = \int \sqrt{\frac{a-x}{a+x}} dx$ | (b) $I = \int \sqrt{\frac{1-x}{1+x}} dx$ |
| Sol.3) | (a) $I = \int \sqrt{\frac{a-x}{a+x}} dx$ | |

| | |
|--------|--|
| | $ \begin{aligned} &= \int \sqrt{\frac{a-x}{a+x}} \times \frac{a-x}{a-x} dx \\ &= \int \frac{a-x}{\sqrt{a^2-x^2}} dx \end{aligned} $ <p>Separate</p> $ \begin{aligned} &= a \int \frac{1}{\sqrt{a^2-x^2}} dx - \int \frac{x}{\sqrt{a^2-x^2}} dx \\ &\text{put } a^2 - x^2 = t \\ &-2x dx = dt \\ &x dx = \frac{dt}{2} \\ &= a \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\ &= a \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2} \times 2\sqrt{t} + c \\ &= a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2} + c \quad \text{ans.} \end{aligned} $ |
| → | <u>Type: Divide by $\cos^2 x$</u> |
| Q.4) | $(a) I = \int \frac{1}{1+3\sin^2 x+8\cos^2 x} dx$ $(b) I = \int \frac{1}{3+\sin(2x)} dx$ |
| Sol.4) | $(a) I = \int \frac{1}{1+3\sin^2 x+8\cos^2 x} dx$ <p>Divide by $\cos^2 x$</p> $ \begin{aligned} I &= \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x + 8} dx \\ &= \int \frac{\sec^2 x dx}{(1+\tan^2 x)+3\tan^2 x+8} \end{aligned} $ <p>put $\tan x = t$</p> $ \begin{aligned} \sec^2 x dx &= dt \\ I &= \int \frac{dt}{(1+t^2)+3t^2+8} \\ &= \int \frac{dt}{4t^2+9} \\ &= \frac{1}{4} \int \frac{1}{t^2+(\frac{3}{2})^2} dt \\ &= \frac{1}{4} \times \frac{2}{3} \tan^{-1}\left(\frac{2t}{3}\right) + c \\ I &= \frac{1}{6} \tan^{-1}\left(\frac{2\tan x}{3}\right) + c \quad \text{ans.} \end{aligned} $ $ \begin{aligned} (b) I &= \int \frac{1}{3+\sin(2x)} dx \\ &= \int \frac{1}{3+2\sin x \cos x} dx \end{aligned} $ <p>Divide N & D by $\cos^2 x$</p> $ \begin{aligned} &= \int \frac{\sec^2 x}{3\sec^2 x + 2\tan x} dx \end{aligned} $ |



| | |
|--------|---|
| | $= \int \frac{\sec^2 x}{3(1+\tan^2 x) + 2\tan x} dx$ <p>put $\tan x = t$ $\sec^2 x dx = dt$ $I = \int \frac{dt}{3(1+t^2)2t}$ $= \int \frac{dt}{3t^2+2t+3}$</p> <p>perfect square: Proceed Your self $\frac{1}{2\sqrt{2}} \left(\frac{3\tan x + 1}{2\sqrt{2}} \right) + c$ ans.</p> |
| Q.5) | $I = \int \frac{1}{\cos(2x)+3\sin^2 x} dx$ |
| Sol.5) | $I = \int \frac{1}{\cos(2x)+3\sin^2 x} dx$ $= \int \frac{1}{\cos^2 x - \sin^2 x + 3\sin^2 x} dx$ $= \int \frac{1}{\cos^2 x + 2\sin^2 x} dx$ <p>Divide N & D by $\cos^2 x$</p> $\int \frac{\sec^2 x}{1+2\tan^2 x}$ <p>put $\tan x = t$ $\sec^2 x dx = dt$</p> $I = \int \frac{dt}{1+2t^2}$ $= \frac{1}{2} \int \frac{1}{(\frac{1}{\sqrt{2}})^2 + t^2} dt$ $= \frac{1}{2} \times \sqrt{2} \tan^{-1}(\sqrt{2}t) + c$ $I = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\tan x) + c$ ans. |
| Q.6) | $I = \int \frac{\sin x}{\sin(3x)} dx$ |
| Sol.6) | $I = \int \frac{\sin x}{\sin(3x)} dx$ $= \int \frac{\sin x}{3\sin x - 4\sin^3 x} dx$ $= \int \frac{\sin x}{\sin x(3-4\sin^2 x)} dx$ $= \int \frac{1}{3-4\sin^2 x} dx$ <p>Divide by $\cos^2 x$</p> $I = \int \frac{\sec^2 x}{3(1+\tan^2 x) - 4\tan^2 x} dx$ |

| | |
|--------|---|
| | $\text{put } \tan x = t$ $\sec^2 x dx = dt$ $I = \int \frac{dt}{3(1+t^2) - 4t^2}$ $I = \int \frac{dt}{3-t^2}$ $I = \frac{1}{2\sqrt{3}} \log \left \frac{\sqrt{3}+t}{\sqrt{3}-t} \right + c$ $= \frac{1}{2\sqrt{3}} \log \left \frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x} \right + c \quad \text{ans.}$ |
| Q.7) | $I = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$ |
| Sol.7) | $I = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$ Divide N & D by $\cos^4 x$ $I = \int \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{\tan^4 x + 1}{\cos^4 x}} dx$ $= \int \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$ $\text{put } \tan^2 x = t$ $\sec^2 x dx = dt$ $I = \frac{1}{2} \int \frac{dt}{t^2 + 1}$ $I = \frac{1}{2} \tan^{-1}(t) + c$ $I = \frac{1}{2} \tan^{-1}(\tan^2 x) + c \quad \text{ans.}$ |
| → | <u>Type: Single Sin x , Cos x</u> |
| Q.8) | (a) $I = \int \frac{1}{1+\sin x + \cos x} dx$ (b) $I = \int \frac{1}{\sin x - \sqrt{3}\cos x} dx$ |
| Sol.8) | $(a) I = \int \frac{1}{1+\sin x + \cos x} dx$ $= \int \frac{1}{1+\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} + \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}} dx$ $= \int \frac{1+\tan^2 \left(\frac{x}{2}\right)}{1+\tan^2 \left(\frac{x}{2}\right) + 2\tan \left(\frac{x}{2}\right) + 1 - \tan^2 \left(\frac{x}{2}\right)} dx$ $= \int \frac{\sec^2 \left(\frac{x}{2}\right)}{2+2\tan \left(\frac{x}{2}\right)} dx$ $\text{put } \tan \frac{x}{2} = t$ $\therefore \sec^2 \left(\frac{x}{2}\right) \cdot \frac{1}{2} dx = dt$ |

$$\sec^2\left(\frac{x}{2}\right) dx = 2dt$$

$$\therefore I = 2 \int \frac{dt}{2+2t}$$

$$= \frac{1}{2} \log |2+2t| + c$$

$$= \log |2+2\tan\left(\frac{x}{2}\right)| + c \quad \text{ans.}$$

$$(b) I = \int \frac{1}{\sin x - \sqrt{3}\cos x} dx$$

$$= \int \frac{1}{\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} - \sqrt{3}(1-\tan^2\left(\frac{x}{2}\right))} dx$$

$$= \int \frac{1+\tan^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right) - \sqrt{3} + \sqrt{3}\tan^2\left(\frac{x}{2}\right)} dx$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right) - \sqrt{3} + \sqrt{3}\tan^2\left(\frac{x}{2}\right)} dx$$

put $\tan\left(\frac{x}{2}\right) = t$

$$\sec^2\left(\frac{x}{2}\right) dx = 2 dt$$

$$\therefore I = 2 \int \frac{dt}{\sqrt{3}t^2 + 2t - \sqrt{3}}$$

$$I = \frac{2}{\sqrt{3}} \int \frac{dt}{t^2 + \frac{2}{\sqrt{3}}t - 1}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{\left(t + \frac{1}{\sqrt{3}}\right)^2 - \frac{4}{3}}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{\left(t + \frac{1}{\sqrt{3}}\right)^2 - \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \times \frac{1}{2 \times \frac{2}{\sqrt{3}}} \log \left| \frac{t + \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}}{t + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{3}t - 1}{\sqrt{3}t + 3} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{3}\tan\left(\frac{x}{2}\right) - 1}{\sqrt{3}\tan\left(\frac{x}{2}\right) + 3} \right| + c \quad \text{ans.}$$

→ Type: $\int \frac{asinx+b\cosx}{csinx+d\cosx} dx$

Q.10) $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} dx$

Sol.10) $I = \int \frac{3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$
let $3 \sin x + 2 \cos x = A \left[\frac{d}{dx} (3 \cos x + 2 \sin x) \right] + B[3\cos x + 2 \sin x]$
 $3 \sin x + 2 \cos x = A(-3 \sin x + 2 \cos x) + B(3 \cos x + 2 \sin x)$



equating coefficients of $\sin x$ and $\cos x$ on both sides

$$3 = 1 - 3A + 2B$$

$$2 = 2A + 3B$$

solving these two equations, we get

$$A = -\frac{5}{13} \text{ and } B = \frac{12}{13}$$

$$\therefore I = \int \frac{\frac{-5}{13}(-3 \sin x + 2 \cos x) + \frac{12}{13}(3 \cos x + 2 \sin x)}{3 \cos x + 2 \sin x} + c$$

Separate

$$I = \frac{-5}{13} \int \frac{-3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx + \frac{12}{13} \int dx$$

put $3 \cos x + 2 \sin x = t$

$$(-3 \sin x + 2 \cos x)dx = dt$$

$$\therefore I = \frac{-5}{13} \int \frac{dt}{t} + \frac{12}{13}x$$

$$I = \frac{-5}{13} \log |3 \cos x + 2 \sin x| + \frac{12}{13}x + c \quad \text{ans.}$$