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<u>Integration</u> (Indefinite Integrals)

Q.1) (i) $I = \int \frac{x^4+1}{x^2+1} dx$ (ii)$I = \int \frac{x^3+x}{\sqrt{x^4+1}} dx$

Sol.1) (i) $I = \int \frac{x^4+1}{x^2+1} dx$

Degree of N^r > degree of D^r

= $\int \theta + \frac{R}{D} dx$

= $x^2 - 1 + \frac{2}{x^2+1} dx$

$I = \frac{x^3}{3} - x + 2\tan^{-1}x + c$

$$
\begin{array}{r|l}
 & x^2 - 1 \\
\hline
x^2 + 1 & x^4 + 1 \\
 & -(x^2 + x^2) \\
\cline{2-2}
 & -x^2 + 1 \\
 & -(-x^2 + 1) \\
\cline{2-2}
 & \quad 2 \\
\cline{2-2}
\end{array}
$$

(ii)$I = \int \frac{x^3+x}{\sqrt{x^4+1}} dx$

= $\int \frac{x^3}{x^4+1} dx + \int \frac{x}{x^4+1} dx$

put $x^4 + 1 = t$in (I) and put $x^2 = z$in (II)

$4x^3 dx = dt$ $\qquad$ $2xdx = dz$

$x^3 dx = \frac{dt}{4}$ $\qquad$ $xdx = \frac{dz}{2}$

$\therefore I = \frac{1}{4}\int \frac{dt}{t} + \frac{1}{2}\int \frac{dz}{z^2+1}$

= $\frac{1}{4}\log \mid t \mid + \frac{1}{2}\tan^{-1}(z) + c$

= $\frac{1}{4}\log \mid x^4 + 1 \mid + \frac{1}{2}\tan^{-1}(x^2) + c$ ans.

→ **<u>Type</u>:** $\int \frac{1}{Quadratic} dx$**Make Perfect Square**

Q.2) (i)$I = \int \frac{1}{2x^2+x-1} dx$ (ii)$I = \int \frac{1}{3+2x-x^2} dx$

Sol.2) (i)$I = \int \frac{1}{2x^2+x-1} dx$

Make perfect square

= $\frac{1}{2}\int \frac{1}{x^2+\frac{x}{2}-\frac{1}{2}} dx$

$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\frac{1}{16}-\frac{1}{2}}dx$

$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\frac{9}{16}}dx$

$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}dx$

$= \frac{1}{2}\times\frac{1}{2\times\frac{3}{4}}\log\left|\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}\right|+c$ ......$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$

$= \frac{1}{3}\log\left|\frac{4x-2}{4x+4}\right|+c$

$I=\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$ ans.

(ii) $I=\int\frac{1}{3+2x-x^2}dx$

Make a perfect square

$= -\int\frac{1}{x^2-2x-3}dx$

$= -\int\frac{1}{(x-1)^2-1-3}dx$

$= -\int\frac{1}{(x-1)^2-(2)^2}dx$

$= \int\frac{1}{(2)^2-(x-1)^2}dx$

$= \frac{1}{2\times2}\log\left|\frac{2+x-1}{2-x+1}\right|+c$ ......$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

$= I=\frac{1}{4}\log\left|\frac{1+x}{3-x}\right|+c$ ans.

Q.3) (a) $I=\int\frac{1}{\sqrt{(x-1)(x-2)}}dx$ (b) $I=\int\frac{1}{\sqrt{7-3x-2x^2}}dx$

(c) $I=\int\frac{1}{\sqrt{(x-a)(x-b)}}dx$

Sol.3) (a) $I=\int\frac{1}{\sqrt{(x-1)(x-2)}}dx$

$I=\int\frac{1}{\sqrt{x^2-3x+2}}dx$

$= \int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2}}dx$

$= \int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$

$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$

$= I=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{(x^2-3x+2)}\right|+c$ ans.

(b) $I=\int\frac{1}{\sqrt{7-3x-2x^2}}dx$

$$= \int \frac{1}{\sqrt{-2\left(x^2+\frac{3}{2}x-\frac{7}{2}\right)}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[\left(x+\frac{3}{4}\right)^2-\frac{9}{16}-\frac{7}{2}\right]}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{65}}{4}\right)^2\right]}}dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{\sqrt{65}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}}dx$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{65}}{4}}\right)+c \qquad \ldots\ldots\ldots\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{65}}\right)+c \quad \text{ans.}$$

(c) $I = \int \frac{1}{\sqrt{(x-a)(x-b)}}dx$

$$= \int \frac{1}{\sqrt{x^2-ax-bx+ab}}dx$$

$$= \int \frac{1}{\sqrt{x^2-(a+b)x+ab}}dx$$

$$= \int \frac{1}{\sqrt{\left\{x-\frac{(a+b)}{2}\right\}^2-\left(\frac{a+b}{2}\right)^2+ab}}dx$$

$$= \int \frac{1}{\sqrt{\left\{x-\frac{(a+b)}{2}\right\}^2-\left\{\frac{(a+b)^2}{4}-ab\right\}}}dx$$

$$= \int \frac{1}{\left\{x-\frac{(a+b)}{2}\right\}^2-\left\{\frac{a^2+b^2+2ab-4ab}{4}\right\}}dx$$

$$= \int \frac{1}{\left\{x-\frac{(a+b)}{2}\right\}^2-\left\{\frac{(a-b)}{2}\right\}^2}dx$$

$$= \log\left|\left(x-\frac{(a+b)}{2}\right)+\sqrt{\left\{x-\frac{a+b}{2}\right\}^2-\left(\frac{a-b}{2}\right)^2}\right|+c$$

$$= \log\left|\left(x-\frac{(a+b)}{2}\right)+\sqrt{(x-a)(x-b)}\right|+c \qquad \text{ans.}$$

→ **Type: Separate** $\int \frac{Linear}{SpecialIntegral}dx$

Q.4) (a) $I = \int \frac{3x-1}{x^2+4}dx$ (b) $I = \int \frac{3-2x}{\sqrt{x^2+4}}dx$

Sol.4) (a) $I = \int \frac{3x-1}{x^2+4}dx$

Separate

$$= 3\int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

put $x^4 + 4 = tin$ (I)

	$2xdx = dt$ $xdx = \frac{dt}{2}$ $\therefore I = 3 \times \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ $= \frac{3}{2}\log \mid t \mid - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ $I = \frac{3}{2}\log \mid x^2 + 4 \mid - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ ans. (b)$I = \int \frac{3-2x}{\sqrt{x^2+4}}dx$ $= 3\int \frac{1}{\sqrt{x^2+4}}dx - 2\int \frac{x}{\sqrt{x^2+4}}dx$ put $x^4 + 4 = t$ in (II) $2\,x\,dx = dt$ $x\,dx = \frac{dt}{2}$ $I = 3\log \mid x + \sqrt{x^2+4} \mid - \frac{2}{2}\int \frac{dt}{\sqrt{t}}$ $= 3\log \mid x + \sqrt{x^2+4} \mid - 2\sqrt{t} + c$ $= 3\log \mid x + \sqrt{x^2+4} \mid - 2\sqrt{x^2+4}$ ans.
→	**Type: Substitution and then** $\int \frac{1}{Qua}dx$
Q.5)	(a)$I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5}dx$ (b)$I = \int \frac{\sin(2x)}{\sqrt{\cos^4 x - \sin^2 x + 2}}dx$
Sol.5)	(a)$I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5}dx$ put $\sin x = t$ $\cos x\,dx = dt$ $\therefore I = \int \frac{dt}{t^2+4t+5}dx$ $= \int \frac{1}{(t+2)^2-4+5}dt$ $= \int \frac{1}{(t+2)^2+1}dt$ $= \tan^{-1}(t+2) + c$ $= I = \tan^{-1}(\sin x + 2) + c$ ans. (b)$I = \int \frac{\sin(2x)}{\sqrt{\cos^4 x - \sin^2 x + 2}}dx$ $= \int \frac{\sin(2x)}{\sqrt{\cos^4 x - (1-\cos^2 x) + 2}}dx$ put $\cos x = t$ $-2\cos x \sin x\,dx = dt$

	$\sin(2x)dx = dt$ $\therefore I = \int \frac{dt}{\sqrt{t^2-(1-t)+2}} dt$ $= -\int \frac{1}{\sqrt{t^2+t+1}} dt$ Proceed Yourself $-\log\left|\left(\cos^2 x + \frac{1}{2}\right) + \sqrt{\cos^4 x + \cos^2 x + 1}\right| + c$ ans.
Q.6)	(a) $I = \int \sqrt{\sec x - 1} dx$
Sol.6)	(a) $I = \int \sqrt{\sec x - 1} dx$ $= \int \sqrt{\frac{1}{\cos x} - 1} dx$ $= \int \sqrt{\frac{1-\cos x}{\cos x}} dx$ Rationalize $I = \int \sqrt{\frac{(1-\cos x)(1+\cos x)}{\cos x(1+\cos x)}} dx$ $= \int \sqrt{\frac{\sin^2 x}{\cos x+\cos^2 x}} dx$ $I = \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}} dx$ put $\cos x = t$ $\sin x\, dx = -dt$ $I = -\int \frac{dt}{t^2+t}$ Proceed Yourself $-\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + c$ ans.
Q.7)	(a) $I = \int \sqrt{\frac{x}{a^3-x^3}} dx$ (b) $I = \int \frac{1}{x^{2/3}\sqrt{x^{2/3}-4}} dx$
Sol.7)	(a) $I = \int \sqrt{\frac{x}{a^3-x^3}} dx$ $= \int \frac{\sqrt{x}}{a^3-x^3} dx$ $= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}} dx$ put $x^{3/2} = t$ $\frac{3}{2}x^{1/2}dx = dt$ $\sqrt{x}dx = \frac{2}{3}dt$

	$\therefore I = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$ $= \frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c$ $= \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c$ ans. (b)$I = \int \frac{1}{x^{2/3}\sqrt{x^{2/3}-4}}dx$ $I = \int \frac{1}{x^{2/3}\sqrt{(x^{1/3})^2 - (2)^2}}dx$ put $x^{1/3} = t$ $\frac{1}{3}x^{-2/3}dx = dt$ $\frac{1}{x^{2/3}}dx = 3dt$ $\therefore I = 3\int \frac{dt}{\sqrt{t^2-2^2}}$ $= 3\log \mid t + \sqrt{t^2-4} \mid +c$ $= 3\log \mid x^{1/3} + \sqrt{x^{2/3}-4} \mid +c$ ans.
Q.8)	(a)$I = \int \frac{1}{e^x+1}dx$ (b)$I = \int \frac{1}{\sqrt{1-e^{2x}}}dx$
Sol.8)	(a)$I = \int \frac{1}{e^x+1}dx$ $= \int \frac{1}{\frac{1}{e^{-x}}+1}dx$ L.C.M $= \int \frac{e^{-x}}{1+e^{-x}}dx$ put $1 + e^{-x} = t$ $-e^{-x}dx = dt$ $e^{-x}dx = -dt$ $I = -\int \frac{dt}{t}$ $= -\log \mid 1 + e^{-x} \mid +c$ $= -\log\left\|\frac{e^x+1}{e^x}\right\| + c$ ans. (b)$I = \int \frac{1}{\sqrt{1-e^{2x}}}dx$ $= \int \frac{1}{\sqrt{1-\frac{1}{e^{-2x}}}}dx$ $= \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}}dx$ put $e^{-x} = t$

	$e^{-x}dx = -dt$ $\therefore I = -\int \frac{dt}{\sqrt{t^2-1}}$ $= -\log \mid t + \sqrt{t^2-1} \mid +c$ $= -\log \mid e^{-x} + \sqrt{e^{-2x}-1} \mid +c$ ans.
Q.9)	(a) $I = \int \frac{\sin(2x)\cos(2x)}{\sqrt{9-\cos^4(2x)}}dx$ (b) $I = \int \frac{\sin x+\cos x}{\sqrt{\sin(2x)}}dx$ (c) $I = \int \frac{\sin x-\cos x}{\sqrt{\sin(2x)}}dx$
Sol.9)	(a) $I = \int \frac{\sin(2x)\cos(2x)}{\sqrt{9-\cos^4(2x)}}dx$ $= \int \frac{\sin(2x)\cos(2x)}{\sqrt{9-(\cos^2(2x))^2}}dx$ put $\cos^2(2x) = t$ $-2\cos(2x).\sin(2x).2dx = dt$ $\cos(2x).\sin(2x) = \frac{-dt}{4}$ $\therefore I = -\frac{1}{4}\int \frac{dt}{\sqrt{3-t^2}}$ $= \frac{-1}{4}\sin^{-1}\left(\frac{t}{3}\right) + c$ $I = -\frac{1}{4}\sin^{-1}\left(\frac{\cos^2(2x)}{3}\right) + c$ ans. (b) $I = \int \frac{\sin x+\cos x}{\sqrt{\sin(2x)}}dx$ $= \int \frac{\sin x+\cos x}{\sqrt{1-1+\sin(2x)}}dx$ $= \int \frac{\sin x+\cos x}{\sqrt{1-(1+\sin 2x)}}dx$ $= \int \frac{\sin x+\cos x}{\sqrt{1-[\sin^2 x+\cos^2 x-2\sin x\cos x]}}dx$ $= \int \frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$ put $\sin x - \cos x = t$ $(\cos x + \sin x)dx = dt$ $\therefore I = \int \frac{dt}{\sqrt{1-t^2}}$ $= \sin^{-1}t + c$ $= \sin^{-1}(\sin x - \cos x) + c$ ans.
→	**Type**: $\int \frac{Linear}{Quadratic}dx$ and $\int \frac{Linear}{\sqrt{Quadratic}}dx$ **Make Derivative of Quadratic in Number by Adjustment**
Q.10)	(a) $I = \int \frac{4x+1}{x^2+3x+2}dx$ (b) $I = \int \frac{3x-2}{1-x-x^2}dx$

Sol.10) (a)$I = \int \frac{4x+1}{x^2+3x+2}dx$

To make $(2x+3)$

$$= 2\int \frac{2x+\frac{1}{2}}{x^2+3x+2}dx$$

$$= 2\int \frac{2x+\frac{1}{2}+3-3}{x^2+3x+2}dx$$

$$= 2\int \frac{(2x+3)-\frac{5}{2}}{x^2+3x+2}dx$$

Separate

$$I = 2\int \frac{2x+3}{x^2+3x_2}dx - 2\times\frac{5}{2}\int \frac{1}{x^2+3x+2}dx$$

put $I = 2\int \frac{2x+3}{x^2+3x+2}dx - 2\times\frac{5}{2}\int \frac{1}{x^2+3x+2}dx$ in (I)

$(2x+3)dx = dt$

$$\therefore I = 2\int \frac{dt}{t} - 5\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+2}dx$$

$$= 2\log\mid t\mid -5\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$= 2\log\mid x^2+3x+2\mid -5\times\frac{1}{2\times\frac{1}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|+c$$

$$= 2\log\mid x^2+3x+2\mid -5\log\left|\frac{2x+2}{2x+4}\right|+c$$ ans.

(b)$I = \int \frac{3x-2}{1-x-x^2}dx$

To make $(-2x-1)$

$$= 3\int \frac{x-\frac{2}{3}}{1-x-x^2}dx$$

$$= \frac{-3}{2}\int \frac{-2x+\frac{4}{3}}{1-x-x^2}dx$$

$$= \frac{-3}{2}\int \frac{-2x+\frac{4}{3}-1+1}{1-x-x^2}dx$$

$$= \frac{-3}{2}\int \frac{(-2x-1)+\frac{7}{3}}{1-x-x^2}dx$$

$$= \frac{-3}{2}\int \frac{(-2x-1)}{1-x-x^2}dx - \frac{3}{2}\times\frac{7}{3}\int \frac{1}{1-x-x^2}dx$$

put $1-x-x^2 = t$

$(-2x-1)dx = dt$

$$\therefore I = -\frac{3}{2}\int \frac{dt}{t} - 7\int \frac{1}{1-x-x^2}dx$$

$$= \frac{-3}{2}\log\mid t\mid -7\int \frac{1}{-(x^2+x-1)}dx$$

$$= \frac{-3}{2}\log\mid 1-x-x^2\mid -7\int \frac{1}{-\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-1\right]}dx$$

	$= \frac{-3}{2}\log\mid 1-x-x^2\mid -7\int \frac{1}{-\left[\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2\right]}dx$ $= \frac{-3}{2}\log\mid 1-x-x^2\mid -7\int \frac{1}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}dx$ $= \frac{-3}{2}\log\mid 1-x-x^2\mid -7\times\frac{1}{2\times\frac{\sqrt{5}}{2}}\log\left\|\frac{\frac{\sqrt{5}}{2}+x+\frac{1}{2}}{\frac{\sqrt{5}}{2}-x-\frac{1}{2}}\right\|+c$ $= \frac{-3}{2}\log\mid 1-x-x^2\mid -\frac{7}{\sqrt{5}}\log\left\|\frac{\sqrt{5}+1+2x}{\sqrt{5}-1-2x}\right\|+c$ ans.