Integration (Indefinite Integrals)

Q.1)	(a) $I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$ (b) $I = \int \frac{\sin x}{\sin(4x)} dx$
Sol.1)	(a) $I = \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$ since degree of N = degree, D = Divide $\therefore I = \int 1 + \frac{-4x^2-10}{(x^2+3)(x^2+4)} dx$ $I = x - \int \frac{4x^2+10}{(x^2+3)(x^2+4)} dx$ let $x^2 = y$ $\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)} = \frac{4y+10}{(y+3)(y+4)}$ let $\frac{4y+10}{(y+3)(y+4)} = \frac{A}{y+3} + \frac{B}{y+4}$ $4y + 10 = A(y+4) + B(y+3)$ Comp. the coefficient of y and constant $4 = A + B$ $10 = 4A + 3B$ solving these equation, we get $A = -2$ $B = 6$ $\therefore I = x - \int \frac{-2}{x^2+3} + \frac{6}{x^2+4} dx$ $I = x + 2\int \frac{1}{x^2+(\sqrt{3})^2} dx - 6\int \frac{1}{x^2+2^2} dx$ $I = x + 2 \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ ans. (b) $I = \int \frac{\sin x}{\sin(4x)} dx$ $= \int \frac{\sin x}{2\sin(2x)\cos(2x)} dx$ $= \int \frac{\sin x}{4\sin x.\cos x.\cos(2x)} dx$ $= \frac{1}{4}\int \frac{1}{\cos x.(1-2\sin^2 x)} dx$ ......$\{\cos(2\theta) = 1 - 2\sin^2\theta\}$ multiply and divide by cos x $= \frac{1}{4}\int \frac{\cos x}{\cos^2 x(1-2\sin^2 x)} dx$ $= \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)} dx$ put $\sin x = t$ $\therefore \cos x\, dx = dt$ $\therefore I = \frac{1}{4}\int \frac{1}{(1-t^2)(1-2t^2)}$ let $t^2 = y$

$\therefore \frac{1}{(1-t^2)(1-2t^2)} = \frac{1}{(1-y)(1-2y)}$

let $(1-y)(1-2y) = \frac{A}{1-y} + \frac{B}{1-2y}$

$1 = A(1-2y) + B(1-y)$

$0 = -2A - B$

$1 = A + B$

$1 = -A$

$\therefore \ A = -1 \ and \ B = 2$

$\therefore I = \frac{1}{4}\int \frac{-1}{1-y} + \frac{2}{1-2y} dt$

$= \frac{1}{4}\int \frac{-1}{1-t^2} + \frac{2}{1-2t^2} dt$

$= \frac{-1}{4}\int \frac{1}{1-t^2} dt + \frac{1}{2}\int \frac{1}{1-2t^2} dt$

$= \frac{-1}{4}\int \frac{1}{1-t^2} dt + \frac{1}{4}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 - t^2}$

$= \frac{-1}{4} \times \frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + 1 \times \frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+t}{\frac{1}{\sqrt{2}}-t}\right| + c$

$I = \frac{-1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + c$ ans.

→ **x^4 type**

form : $\int \frac{x^2+1}{x^4+1} dx$ **(.) D^r → x^4** **(.) No old power of x**

$\int \frac{x^2-1}{x^4+1} dx$ **(.)D → constant term (+ve)** **(.) D^r constant = (N^r constant)^2**

Procedure :

Divide N & D by x^2

∴ In N^r we get either $1 + \frac{1}{x^2} dx$ **Or** $\frac{1}{x^2} dx$

In D^r put $x - \frac{1}{x} = t$**(or)** $x + \frac{1}{x} = t$

$a^2 + b^2 = (a+b)^2 - 2ab$

$2 + b^2 = (a-b)^2 2ab$

Q.2) (a)$I = \int \frac{x^2+1}{x^4+1} dx$ (b)$I = \int \frac{x^2-4}{x^4-16} dx$

Sol.2) (a)$I = \int \frac{x^2+1}{x^4+1} dx$

Divide N & D by x^2

$\therefore I = \int \frac{1+1/x^2}{x^2+\frac{1}{x^2}} dx$

$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2} dx$

put $\frac{1}{x} = t$

	$\left(1+\frac{1}{x^2}\right)dx = dt$ $I = \int \frac{dt}{t^2+2}$ $= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c$ $= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+c$ $I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+c$ ans. (b)$I = \int \frac{x^2-4}{x^4-16}dx$ Divide N & D by x^2 $= \int \frac{1-\frac{4}{x^2}}{x^2-\frac{16}{x^2}}dx$ $= \int \frac{1-\frac{4}{x^2}}{\left(x+\frac{4}{x}\right)^2-8}dx$ put $x+\frac{4}{x}=t$ $\left(1-\frac{4}{x^2}\right)dx = dt$ $\therefore I = \int \frac{dt}{t^2(2\sqrt{2})^2}$ $= \frac{1}{2\times2\sqrt{2}}\log\left\|\frac{t-2\sqrt{2}}{t+2\sqrt{2}}\right\|+c$ $= \frac{1}{4\sqrt{2}}\log\left\|\frac{x+\frac{4}{x}-2\sqrt{2}}{x+\frac{4}{x}+2\sqrt{2}}\right\|+c$ $= \frac{1}{4\sqrt{2}}\log\left\|\frac{x^2-2\sqrt{2}x+4}{x^2+2\sqrt{2}x+4}\right\|+c$ ans.
Q.3)	(a)$I = \int \frac{x^2}{x^4+x^2+1}dx$ (b)$I = \int \frac{1}{x^4+1}dx$
Sol.3)	(a)$I = \int \frac{x^2}{x^4+x^2+1}dx$ (double set) Divide N & D by x^2 $I = \int \frac{1}{x^2+1+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{2}{x^2+\frac{1}{x^2}+1}$ (adjustment) $= \frac{1}{2}\int \frac{1+1+\frac{1}{x^2}-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}$ (adjustment) $= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx + \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx$

	$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}dx + \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+1}dx$ put $x-\frac{1}{x}=t$ put $x+\frac{1}{x}=z$ $\therefore \left(1+\frac{1}{x^2}\right)dx = dt$ $\therefore 1-\frac{1}{x^2}dx = dz$ $\therefore I = \frac{1}{2}\int \frac{dt}{t^2+3} + \frac{1}{2}\int \frac{dz}{z^2-1}$ $= \frac{1}{2}\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + \frac{1}{2}\times\frac{1}{2\times1}\log\left\|\frac{z-1}{z+1}\right\| + c$ $= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right) + \frac{1}{4}\log\left\|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right\| + c$ $= \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) + \frac{1}{4}\log\left\|\frac{x^2-x+1}{x^2+x+1}\right\| + c$ ans. (b) $I = \int \frac{1}{x^4+1}dx$ (double set) Divide N & D by x^2 $I = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{\frac{1}{x^2}+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{\frac{1}{x^2}+\frac{1}{x^2}+1-1}{x^2+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$ $= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$ put $x-\frac{1}{x}=t$ put $x+\frac{1}{x}=z$ $\left(1+\frac{1}{x^2}\right)dx = dt$ $\left(1-\frac{1}{x^2}\right)dx = dz$ $\therefore I = \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{dz}{z^2-2}$ $= \frac{1}{2}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\times\sqrt{2}}\log\left\|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right\| + c$ $= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left\|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right\| + c$ $= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left\|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right\| + c$ ans.
Q.4)	(a) $I = \int \sqrt{\tan\theta} + \sqrt{\cot\theta}\, d\theta$ (b) $I = \int \sqrt{\cot\theta}\, d\theta$

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Sol.4)

(a) $I = \int \sqrt{\tan\theta} + \sqrt{\cot\theta}\, d\theta$

$= \int \sqrt{\tan\theta} + \frac{1}{\sqrt{\tan\theta}} d\theta$

$= \int \frac{\tan\theta+1}{\sqrt{\tan\theta}} dx$

put $\tan\theta = t^2$

$sec^2\theta . d\theta = 2t\, dt$

$d\theta = \frac{2t\, dt}{sec^2\theta}$

$d\theta = \frac{2t\, dt}{\tan^2\theta+1}$

$d\theta = \frac{2t\, dt}{t^4+1}$ ......$\{\because \tan\theta = t^2\}$

$\therefore I = \int \frac{t^2+1}{t} . \frac{2tdt}{t^4+1}$

$= I = 2\frac{\int t^2+1}{t^4+1} dt$ (single set)

Proceed yourself

$I = \sqrt{2}\tan^{-1}\left(\frac{\tan\theta-1}{\sqrt{2\tan\theta}}\right) + c$ ans.

(b) $I = \int \sqrt{\cot\theta}\, d\theta$ (double set)

put $\cot\theta = t^2$

$\therefore -cosce^2\theta . d\theta = 2t\, dt$

$d\theta = \frac{-2tdt}{cosec^2\theta}$

$d\theta = \frac{-2tdt}{\cot^2\theta+1}$

$d\theta = \frac{-2tdt}{t^4+1}$ ......$\{\because \cot\theta = t^2\}$

$\therefore I = \int t . \left(\frac{-2tdt}{t^4+1}\right)$

$= -2\int \frac{t^2}{t^4+1} dt$

Divide N & D by t^2

$= -2\int \frac{1}{t^2+\frac{1}{t^2}} dt$

$= \frac{-2}{2} \int \frac{2}{t^2+\frac{1}{t^2}} dt$

$= -\int \frac{1+1}{t^2+\frac{1}{t^2}} dt$

$= -\int \frac{1+1+\frac{1}{t^2}-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt$

$= -\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt - \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt$

$= -\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t^2}\right)+2} dx - \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2} dt$

	put $t-\frac{1}{t}=u$ and put $t+\frac{1}{t}=v$ $\left(1+\frac{1}{t^2}\right)dt=du \qquad \left(1-\frac{1}{t^2}\right)dt=dv$ $\therefore I=-\int\frac{du}{u^2+(\sqrt{2})^2}-\int\frac{dv}{v^2-(\sqrt{2})^2}$ $=\frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)-\frac{1}{2\times\sqrt{2}}\log\left\|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right\|+c$ $=\frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)-\frac{1}{2\sqrt{2}}\log\left\|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right\|+c$ $=\frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)-\frac{1}{2\sqrt{2}}\log\left\|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right\|+c$ replacing t by $\sqrt{\cot\theta}$ $I=\frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta-1}{\sqrt{2\cot\theta}}\right)-\frac{1}{2\sqrt{2}}\log\left\|\frac{\cot\theta-\sqrt{2cost\theta}+1}{\cot\theta+\sqrt{2\cot\theta}+1}\right\|+c$ ans.
Q.5)	$I=\int\frac{1}{\sin^4x+\cos^4x}dx$
Sol.5)	$I=\int\frac{1}{\sin^4x+\cos^4x}dx$ Divide N & D by $\cos^4x$ $I=\int\frac{sec^4x}{\tan^4x+1}dx$ $=\int\frac{sec^2x.sec^2x}{\tan^4x+1}dx$ $=\int\frac{(\tan^2x+1).sec^2x}{\tan^4x+1}dx$ put $tan\,x=t$ $\therefore$ $\sec^2$ x dx = dt $I=\int\frac{t^2+1}{t^4+1}dt$ (single set) Proceed Yourself $I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2x-1}{\sqrt{2}\tan x}\right)+c$ ans.
→	**Type** : **(.)** $\int\frac{\phi(x)}{linear\sqrt{linear}}$; put $Linear=t^2$ **(.)** $\int\frac{\phi(x)}{equadratic\sqrt{linear}}$; put $Linear=t^2$ **(.)** $\int\frac{\phi(x)}{Linear\sqrt{equadratic}}$; put $Linear=\frac{1}{t}$
Q.6)	(a) $I=\int\frac{1}{(x-3)\sqrt{x+1}}dx$ (b) $I=\int\frac{1}{(x-1)\sqrt{2x+3}}dx$
Sol.6)	(a) $I=\int\frac{1}{(x-3)\sqrt{x+1}}dx$ put $x+1=t^2$ $dx=2t\,dt$

	$\therefore I = \int \frac{1}{(x-3)t}.2t\, dt$ $= 2\int \frac{1}{(t^2-1-3)} dt$ ...{∵ $x = t^2 - 1$} $= 2\int \frac{1}{t^2-4} dt$ $= 2 \times \frac{1}{2\times 2} \log\left\|\frac{t-2}{t+2}\right\| + c$ $I = \frac{1}{2}\log\left\|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right\| + c$ (b)$I = \int \frac{1}{(x-1)\sqrt{2x+3}} dx$ put $2x + 3 = t^2$ $2\, dx = 2t\, dt$ $dx = t\, dt$ $\therefore I = \int \frac{t}{(x-1).t} dt$ $= \int \frac{dt}{\left(\frac{t^2-3}{3}-1\right)}$ $= 2\int \frac{1}{t^2-5} dt$ $= 2 \times \frac{1}{2\sqrt{5}} \log\left\|\frac{t-\sqrt{5}}{t+\sqrt{5}}\right\| + c$ $= \frac{1}{\sqrt{5}} \log\left\|\frac{\sqrt{2x+3}-\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}}\right\| + c$ ans.
Q.7)	(a)$I = \int \frac{1}{(x^2-4)\sqrt{x+1}} dx$ (b)$I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$
Sol.7)	(a)$I = \int \frac{1}{(x^2-4)\sqrt{x+1}} dx$ put $x + 1 = t^2$ $dx = 2t\, dt$ $\therefore I = \int \frac{2t}{(x^2-4)t} dt$ $= 2\int \frac{1}{(t^2-1)^2-4} dt$ $= 2\int \frac{1}{t^4-2t^2+1-4} dt$ $= 2\int \frac{1}{t^4-2t^2-3} dt$ $= 2\int \frac{1}{(t^2-3)(t^2+1)}$ Partial fraction: type 4 let $t^2 = y$ Proceed yourself $I = \frac{1}{4\sqrt{3}} \log\left\|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right\| - \frac{1}{2}\tan^{-1}(\sqrt{x+1}) + c$ ans. (b)$I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$

	put $x + 1 = t^2$ $dx = 2t\,dt$ $\therefore I = \int \frac{x+2}{(x^2+3x+3)} \cdot \frac{2t}{t} dt$ $= 2\int \frac{(t^2-1)+2}{(t^2-1)+3(t^2-1)+3} dt$ ...{∵ $x = t^2 - 1$} $= 2\int \frac{t^2-1}{t^4-2t^2+1+3t^2-3+3} dt$ $= 2\int \frac{t^2-1}{t^4+t^2+1}$ (single set) x^4 type single set Proceed Yourself $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right) + c$ ans.
Q.8)	$I = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$
Sol.8)	$I = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$ put $x + 1 = \frac{1}{t}$ $dx = -\frac{1}{t^2} dt$ $\therefore I = \int \frac{\frac{-1}{t^2}}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2 - 1}}$ $= -\int \frac{1}{t\sqrt{\frac{1}{t^2} - \frac{2}{t} + 1 - 1}} dt$ $= -\int \frac{1}{t\sqrt{\frac{1-2t}{t^2}}} dt$ $= -\int \frac{1}{t\frac{\sqrt{1-2t}}{t}} dt$ $= -\int \frac{1}{\sqrt{1-2t}} dt$ $= \frac{-2\sqrt{1-2t}}{-2} + c$ $= \sqrt{1-2t} + c$ $= \sqrt{\frac{1-2}{x+1}} + c$ $= \sqrt{\frac{x-1}{x+c}} + c$ ans.

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