

Integration (Indefinite Integrals)

<span style="color: #0070C0;">→</span> <b>Type:</b> (.) $I = \int e^{ax} \sin(bx + c) dx$ $I = \int e^{ax} \cos(bx + c) dx$ <b>I repeats of the two types by Parts</b>		
<b>Q.1)</b> (a) $I = \int e^{2x} \cdot \cos(3x) dx$	(b) $I = \int e^{ax} \cdot \sin(dx + c) dx$	
<b>Sol.1)</b> $(a) I = \int e^{2x} \cdot \cos(3x) dx$ $= \cos(3x) \cdot \frac{e^{2x}}{2} - \int (-3\sin(3x)) \cdot \frac{e^{2x}}{2} dx$ $= \frac{e^{2x}}{2} \cdot \cos(3x) + \frac{3}{2} \int \sin(3x) \cdot e^{2x} dx$ $= \frac{e^{2x}}{2} \cdot \cos(3x) + \frac{3}{2} \left[ \sin(3x) \cdot \frac{e^{2x}}{2} - \int 3\cos(3x) \cdot \frac{e^{2x}}{2} dx \right]$ $I = \frac{e^{2x}}{2} \cdot \cos(3x) + \frac{3}{2} \left[ \frac{e^{2x}}{2} \cdot \sin(3x) - \frac{3}{2} I \right]$ $I = \frac{e^{2x}}{2} \cdot \cos(3x) + \frac{3}{4} e^{2x} \cdot \sin(3x) - \frac{9}{4} I$ $I + \frac{9}{4} I = \frac{e^{2x}}{4} [2\cos(3x) + 3\sin(3x)]$ $\frac{13I}{4} = \frac{e^{2x}}{4} [2\cos(3x) + 3\sin(3x)] + c$ $\therefore I = \frac{e^{2x}}{13} [2\cos(3x) + 3\sin(3x)] + c \text{ ans.}$		
<b>(b)</b> $I = \int e^{ax} \cdot \sin(bx + c) dx$	$= \sin(bx + c) \cdot \frac{e^{ax}}{a} - \int b\cos(bx + c) \cdot \frac{e^{ax}}{a} dx$ $= \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a} \int e^{ax} \cdot \cos(bx + c) dx$ $= \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a} \left[ \cos(bx + c) \cdot \frac{e^{ax}}{a} - \int (-b\sin(bx + c)) \cdot \frac{e^{ax}}{a} dx \right]$ $= \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a} \left[ \frac{e^{ax}}{a} \cdot \cos(bx + c) + \frac{b}{a} \int e^{ax} \cdot \sin(bx + c) dx \right]$ $= \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a} \left[ \frac{e^{ax}}{a} \cos(bx + c) + \frac{b}{a} I \right]$ $I = \frac{e^{ax}}{a} \sin(bx + c) - \frac{b}{a^2} e^{ax} \cdot \cos(bx + c) - \frac{b^2}{a^2} I$ $I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} [a \sin(bx + c) - b \cos(bx + c)] + c$ $I \left( \frac{a^2 + b^2}{a^2} \right) = \frac{e^{ax}}{a^2} [a \sin(bx + c) - b \cos(bx + c)] + c$ $\therefore I = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + c) - b \cos(bx + c)] + c \text{ ans.}$	
<b>Q.2)</b> $I = \int e^x \cdot \cos^2 x dx$		



Sol.2)	$  \begin{aligned}  I &= \int e^x \cdot \cos^2 x dx \\  &= \int e^x \cdot \left\{ \frac{1+\cos(2x)}{2} \right\} dx \\  &= \frac{1}{2} \int e^x + e^x \cdot \cos(2x) dx \\  I &= \frac{1}{2} \int e^x dx + \frac{1}{2} \int e^x \cdot \cos(2x) dx \\  I &= \frac{1}{2} \int e^x dx + \frac{1}{2} I_1 \\  \text{where } I &= \int e^x \cdot \cos(2x) dx \\  &= \cos(2x) \cdot e^x - \int -2\sin(2x) \cdot e^x dx \\  &= e^x \cdot \cos(2x) + 2 \int e^x \cdot \sin(2x) dx \\  &= e^x \cdot \cos(2x) + 2[e^x \cdot \sin(2x) - 2 \int \cos(2x) \cdot e^x dx] \\  I_1 &= e^x \cos(2x) + 2e^x \sin(2x) - 4I_1 \\  5I_1 &= e^x [\cos(2x) + 2\sin(2x)] \\  I_1 &= \frac{e^x}{5} [\cos(2x) + 2\sin(2x)] + c \\  \therefore I &= \frac{1}{2} e^x + \frac{1}{2} \left[ \frac{e^x}{5} \cdot (\cos(2x) + 2\sin(2x)) \right] + c \quad \text{ans.}  \end{aligned}  $
→ Type:	$  \begin{aligned}  I &= \int e^x (f(x) + f'(x)) dx \\  I &= \int e^x \cdot f(x) dx + \int e^x \cdot f'(x) dx \\  &= f(x) \cdot e^x - \int f'(x) \cdot e^x dx + \int e^x \cdot f'(x) dx \\  I &= e^x \cdot f(x) + c  \end{aligned}  $
Q.3)	<p>(a) <math>I = \int e^x \left( \frac{2+\sin(2x)}{1+\cos(2x)} \right) dx</math></p> <p>(b) <math>I = \int e^x \left( \frac{1-\sin x}{1-\cos x} \right) dx</math></p> <p>(c) <math>I = \int e^{2x} \left( \frac{1+\sin(2x)}{1+\cos(2x)} \right) dx</math></p>
Sol.3)	<p>(a) <math>I = \int e^x \left( \frac{2+\sin(2x)}{1+\cos(2x)} \right) dx</math></p> $  \begin{aligned}  &= \int e^x \left[ \frac{2+2\sin x \cos x}{2\cos^2 x} \right] dx \\  &= \int e^x \left[ \frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right] dx \\  &= \int e^x (\sec^2 x + \tan x) dx \\  &\quad f'(x) f(x) \\  &= \int e^x \cdot \tan x dx + \int e^x \sec^2 x dx \\  &= \tan x \cdot e^x - \int \sec^2 x \cdot e^x dx + \int e^x \sec^2 x dx \\  &= e^x \cdot \tan x + c \quad \text{ans.}  \end{aligned}  $ <p>(b) <math>I = \int e^x \left( \frac{1-\sin x}{1-\cos x} \right) dx</math></p> $  \begin{aligned}  &= \int e^x \left[ \frac{1-2\sin^2 x \cos^2 x}{2\sin^2 x} \right] dx  \end{aligned}  $

	$  \begin{aligned}  &= \int e^x \left[ \frac{1}{2\sin^2 \frac{x}{2}} - \frac{2\sin \frac{x}{2} \cdot \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right] dx \\  &= \int e^x \left[ \frac{1}{2} \operatorname{cosec}^2 \left( \frac{x}{2} \right) - \cot \left( \frac{x}{2} \right) \right] dx \\  &\quad f'(x)f(x) \\  &= - \int e^x \cdot \cot \frac{x}{2} dx + \frac{1}{2} \int e^x \cdot \operatorname{cosec}^2 \frac{x}{2} dx \\  &= - \left[ \cot \frac{x}{2} \cdot e^x - \int -\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \cdot e^x dx \right] + \frac{1}{2} \int e^x \cdot \operatorname{cosec}^2 \frac{x}{2} dx \\  &= -e^x \cot \frac{x}{2} - \frac{1}{2} \int e^x \cdot \operatorname{cosec}^2 \frac{x}{2} dx + \frac{1}{2} \int e^x \cdot \operatorname{cosec}^2 \frac{x}{2} dx \\  &= I = -e^x \cdot \cot \frac{x}{2} + c \quad \text{ans.}  \end{aligned}  $
(c)	$  \begin{aligned}  I &= \int e^{2x} \left( \frac{1+\sin(2x)}{1+\cos(2x)} \right) dx \\  &= \int e^{2x} \left( \frac{1+2\sin x \cos x}{2\cos^2 x} \right) dx \\  &= \int e^{2x} \cdot \left( \frac{1}{2} \sec^2 x + \tan x \right) dx \\  &= \int e^{2x} \cdot \tan x dx + \frac{1}{2} \int e^{2x} \cdot \sec^2 x dx \\  &= \tan x \cdot \frac{e^{2x}}{2} - \int \sec^2 x \cdot \frac{e^{2x}}{2} dx + \frac{1}{2} \int e^{2x} \cdot \sec^2 x dx \\  I &= \frac{1}{2} e^{2x} \cdot \tan x + c \quad \text{ans.}  \end{aligned}  $
Q.4)	<p>(a) <math>I = \int e^x - \frac{x}{(x+1)^2} dx</math></p> <p>(b) <math>I = \int e^x \left( \frac{x-4}{(x-2)^3} \right) dx</math></p>
Sol.4)	<p>(a) <math>I = \int e^x - \frac{x}{(x+1)^2} dx</math></p> $  \begin{aligned}  &= \int e^x \left[ \frac{x+1-1}{(x+1)^2} \right] dx \\  &= \int e^x \left[ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right] dx \\  &\quad f(x)f'(x) \\  &= \int e^x \cdot \frac{1}{x+1} dx - \int e^x \cdot \frac{1}{(x+1)^2} dx \\  &= \frac{1}{x+1} \cdot e^x + \int \frac{1}{(x+1)^2} \cdot e^x dx - \int \frac{1}{(x+1)^2} \cdot e^x dx \\  I &= e^x \cdot \frac{1}{x+1} + c \quad \text{ans.}  \end{aligned}  $ <p>(b) <math>I = \int e^x \left( \frac{x-4}{(x-2)^3} \right) dx</math></p> $  \begin{aligned}  &= \int e^x \left( \frac{x-4}{(x-2)^3} \right) dx \\  &= \int e^x \left[ \frac{1}{(x-2)^2} - \frac{2}{(x-2)^3} \right] dx \\  &\quad f(x)f'(x)  \end{aligned}  $ <p>Proceed Yourself</p> $  e^x \cdot \frac{1}{(x-2)^2} + c \quad \text{ans.}  $



Q.5)	$I = \int e^x \cdot \frac{(x^2 + 1)}{(x + 1)^2} dx$
Sol.5)	$  \begin{aligned}  I &= \int e^x \cdot \frac{(x^2 + 1)}{(x + 1)^2} dx \\  &= \int e^x \cdot \left[ \frac{x^2 + 1 + 2x - 2x}{(x+1)^2} \right] dx \\  &= \int e^x \left( \frac{x^2 + 1 + 2x}{(x+1)^2} - \frac{2x}{(x+1)^2} \right) dx \\  &= \int e^x \left( 1 - \frac{2x}{(x+1)^2} \right) dx \\  &= \int e^x dx - 2 \int e^x \cdot \frac{x}{(x+1)^2} dx \\  &= e^x - 2 \int e^x \cdot \left[ \frac{x+1-1}{(x+1)^2} \right] dx \\  &= e^x - 2 \int e^x \left( \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx \\  &= e^x - 2 \left[ \int e^x \cdot \frac{1}{x+1} dx - \int e^x \cdot \frac{1}{(x+1)^2} dx \right] \\  &= e^x - 2 \left[ \frac{1}{(x+1)} \cdot e^x + \int \frac{1}{(x+1)^2} \cdot e^x dx - \int e^x \cdot \frac{1}{(x+1)^2} dx \right] \\  &= e^x - 2 \cdot \frac{e^x}{x+1} + c \\  &= e^x \left( 1 - \frac{2}{(x+1)} \right) + c \\  &= e^x \left( \frac{x-1}{x+1} \right) + c \quad \text{ans.}  \end{aligned}  $
Q6)	<p>(a) <math>I = \int \frac{\log x}{(\log x+1)^2} dx</math></p> <p>(b) <math>I = \int \log(\log x) + \frac{1}{(\log x)^2} dx</math></p>
Sol.6)	<p>(a) <math>I = \int \frac{\log x}{(\log x+1)^2} dx</math></p> <p>put <math>\log x = t</math></p> <p><math>x = e^t</math></p> <p><math>dx = e^t dt</math></p> <p><math>\therefore I = \int \frac{t}{(t+1)^2} \cdot e^t dt</math></p> <p><math>= \int e^t \left[ \frac{t+1-1}{(t+1)^2} \right] dt</math></p> <p><math>= \int e^t \left[ \frac{1}{t+1} - \frac{1}{(t+1)^2} \right] dt</math></p> <p><math>= e^t \cdot \frac{1}{t+1} + c</math></p> <p>replacing <math>t</math></p> <p><math>= x \cdot \frac{1}{\log x + 1} + c \quad \text{ans.}</math></p> <p>(b) <math>I = \int \log(\log x) + \frac{1}{(\log x)^2} dx</math></p> <p>put <math>\log x = t</math></p>



	$\begin{aligned}x &= e^t \\dx &= e^t dt \\ \therefore I &= \int \left( \log t + \frac{1}{t^2} \right) \cdot e^t dt\end{aligned}$ <p>adjustment</p> $\begin{aligned}&= \int e^t \left[ \log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right] dt \\&= \int e^t \left( \log t + \frac{1}{t} \right) dt - \int e^t \left( \frac{1}{t} - \frac{1}{t^2} \right) dt \\&= \left[ \int e^t \log t dt + \int e^t \cdot \frac{1}{t} dt \right] - \left[ \int e^t \cdot \frac{1}{t} dt - \int e^t \cdot \frac{1}{t^2} dt \right] \\&= \left[ \log t \cdot e^t - \int \frac{1}{t} \cdot e^t dt + \int e^t \cdot \frac{1}{t} dt \right] - \left[ \frac{1}{t} \cdot e^t + \int \frac{1}{t^2} \cdot e^t dt - \int e^t \cdot \frac{1}{t^2} dt \right] \\&= \log t \cdot e^t - \frac{1}{t} \cdot e^t + c \\&= e^t \left( \log t - \frac{1}{t} \right) + c \\I &= x \left( \log(\log x) - \frac{1}{\log x} \right) + c \quad \text{ans.}\end{aligned}$
Q.7).	(a) $I = \int e^{-\frac{x}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x} dx$ (b) $I = \int e^{2x} (-\sin x + 2 \cos x) dx$
Sol.7)	$\begin{aligned}(a) I &= \int e^{-\frac{x}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x} dx \\&= \int e^{-\frac{x}{2}} \frac{\sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2}}}{2\cos^2 \left( \frac{x}{2} \right)} dx \\&= \int e^{-\frac{x}{2}} \frac{\sqrt{\left( \sin \frac{x}{2} - \cos \frac{x}{2} \right)^2}}{2\cos^2 \left( \frac{x}{2} \right)} dx \\&= \int e^{-\frac{x}{2}} \frac{\left( \sin \frac{x}{2} - \cos \frac{x}{2} \right)}{2\cos^2 \left( \frac{x}{2} \right)} dt \\&= \int e^{-\frac{x}{2}} \left[ \frac{1}{2} \tan \frac{x}{2} \cdot \sec \frac{x}{2} - \frac{1}{2} \sec \left( \frac{x}{2} \right) \right] dx \\&= \frac{-1}{2} \int e^{-\frac{x}{2}} \sec \left( \frac{x}{2} \right) dx + \frac{1}{2} \int e^{-\frac{x}{2}} \sec \frac{x}{2} \cdot \tan \frac{x}{2} dx \\&= \frac{-1}{2} \left[ \sec \frac{x}{2} \cdot e^{-\frac{x}{2}} (-2) - \int \sec \left( \frac{x}{2} \right) \cdot \tan \frac{x}{2} \cdot \left( \frac{1}{2} \right) \cdot e^{-\frac{x}{2}} (-2) dx \right] + \frac{1}{2} \int e^{-\frac{x}{2}} \sec \frac{x}{2} \tan \frac{x}{2} dx \\&= e^{-\frac{x}{2}} \cdot \sec \frac{x}{2} - \frac{1}{2} \int e^{-\frac{x}{2}} \sec \frac{x}{2} \tan \frac{x}{2} dx + \frac{1}{2} \int e^{-\frac{x}{2}} \sec \frac{x}{2} \tan \frac{x}{2} dx \\&= e^{-\frac{x}{2}} \sec \left( \frac{x}{2} \right) + c \quad \text{ans.}\end{aligned}$
→	<b>Type:</b> $\int \sqrt{\text{Quadratic}} \text{ and } \int \text{Linear} \sqrt{\text{Quadratic}}$ <u>Perfect Square, Use Long Formula</u>
Q.8)	(a) $I = \int \sqrt{(x-3)(5-x)} dx$ (b) $I = \int \sqrt{2x^2 + 3x + 4} dx$ (c) $I = \int \sqrt{3 - 2x - 2x^2} dx$

Sol.8)	$  \begin{aligned}  (a) I &= \int \sqrt{(x-3)(5-x)} dx \\  &= \int \sqrt{5x - x^2 - 15 + 3x} dx \\  &= \int \sqrt{-x^2 + 8x - 15} dx \\  &= \int \sqrt{-[x^2 - 8x + 15]} dx \\  &= \int \sqrt{-[x^2 - 8x + 16 + 1]} dx \\  &= \int \sqrt{-[(x-4)^2 + 1]} dx \\  &= \int \sqrt{1^2 - (x-4)^2} dx \\  &= \frac{(x-4)}{2} \sqrt{1 - (x-4)^2} + \frac{1}{2} \sin^{-1} \left( \frac{x-4}{1} \right) + c \\  &= \frac{(x-4)}{2} \sqrt{(x-3)(5-x)} + \frac{1}{2} \sin^{-1}(x-4) + c  \end{aligned}  $ <p style="text-align: right;">ans.</p>
Q.9)	$I = \int \cos x \sqrt{4 - \sin^2 x} dx$
Sol.9)	$  \begin{aligned}  I &= \int \cos x \sqrt{4 - \sin^2 x} dx \\  \text{put } \sin x &= t \\  \therefore \cos x dx &= dt \\  I &= \int \sqrt{4 - t^2} dt \\  &= \frac{1}{2} \sqrt{4 - t^2} + 2 \sin^{-1} \left( \frac{t}{2} \right) + c \\  &= \frac{\sin x}{x} \sqrt{4 - \sin^2 x} + 2 \sin^{-1} \left( \frac{\sin x}{2} \right) + c  \end{aligned}  $ <p style="text-align: right;">ans.</p>
Q.10)	(a) $I = \int (3x-2)\sqrt{x^2+x+1} dx$ (b) $I = \int (4x+1)\sqrt{x^2-x-2} dx$
Sol.10)	$  \begin{aligned}  (a) I &= \int (3x-2)\sqrt{x^2+x+1} dx \\  (\text{take } 2x+1) \\  &= 3 \int \left( x - \frac{2}{3} \right) \sqrt{x^2+x+1} dx \\  &= \frac{3}{2} \int \left( 2x - \frac{4}{3} \right) \sqrt{x^2+x+1} dx \\  &= \frac{3}{2} \int \left( 2x - \frac{4}{3} + 1 - 1 \right) \sqrt{x^2+x+1} dx \\  &= \int \sqrt{5x - x^2 - 15 + 3x} dx \\  &= \frac{3}{2} \int (2x+1)\sqrt{x^2+x+1} dx - \frac{7}{2} \int \sqrt{x^2+x+1} dx \\  \text{put } x^2+x+1 &= t \text{ in I} \\  (2x+1)dx &= dt \\  &= \frac{3}{2} \int \sqrt{t} dt - \frac{7}{2} \int \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1} dx \\  &= \frac{3}{2} \times \frac{2}{3} (\sqrt{t})^{\frac{3}{2}} - \frac{7}{2} \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx  \end{aligned}  $

	$  \begin{aligned}  &= (\sqrt{t})^{\frac{3}{2}} - \frac{7}{2} \left[ \frac{(x+\frac{1}{2})}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{8} \log \left  \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right  \right] \\  &= (\sqrt{x^2 + x + 1})^{\frac{3}{2}} - \frac{7}{2} \left[ \frac{(2x+1)}{4} \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left  \frac{2x+1}{2} + \sqrt{x^2 + x + 1} \right  \right]  \end{aligned}  $
	<p>(b) <math>I = \int (4x+1)\sqrt{x^2-x-2}dx</math></p> $  \begin{aligned}  &= 2\int \left(2x + \frac{1}{2}\right) \sqrt{x^2 - x - 2} dx \\  &= 2\int \left(2x + \frac{1}{2} - 1 + 1\right) \sqrt{x^2 - x - 2} dx \\  &= 2\int \left(2x - 1 + \frac{3}{2}\right) \sqrt{x^2 - x - 2} dx \\  &= 2\int (2x - 1)\sqrt{x^2 - x - 2} dx + 3\int \sqrt{x^2 - x - 2} dx  \end{aligned}  $ <p>put <math>x^2 - x - 2 = t</math></p> $  \begin{aligned}  (2x - 1)dx &= dt \\  \therefore I &= 2\int dt + 3\int \sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 2} dx \\  &= 2 \times \frac{2}{3}(t)^{\frac{3}{2}} + 3\int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx \\  &= \frac{4}{3}(t)^{\frac{3}{2}} + 3 \left[ \frac{x-\frac{1}{2}}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{9}{8} \log \left  \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} \right  \right] \\  &= \frac{4}{3}(x^2 - x - 2)^{\frac{3}{2}} + 3 \left[ \frac{2x-1}{4} \sqrt{x^2 - x - 2} - \frac{9}{8} \log \left  \frac{2x-1}{2} + \sqrt{x^2 - x - 2} \right  \right] \text{ ans.}  \end{aligned}  $