Integration (Indefinite Integrals)

Q.1)	$I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$
Sol.1)	$I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ put $x = t^2$ $dx = 2\,t\,dt$ $\therefore = 2\int \sqrt{\dfrac{1-t}{1+t}}\cdot t\,dt$ rationalize $= 2\int \sqrt{\dfrac{1-t}{1+t}\times\dfrac{1-t}{1-t}}\cdot t\,dt$ $= 2\int \dfrac{(1-t)}{\sqrt{1-t^2}}\cdot t\,dt$ $= 2\int \dfrac{t-t^2}{\sqrt{1-t^2}}\,dt$ $= 2\int \dfrac{t}{\sqrt{1-t^2}}\,dt - 2\int t^2\sqrt{1-t^2}\,dt$ put $1-t^2 = z$ in (I) $-2t\,dt = dz$ $t\,dt = -\dfrac{dz}{2}$ $\therefore I = -\dfrac{2}{2}\int \dfrac{dz}{\sqrt{z}} + 2\int -\dfrac{t^2}{\sqrt{1-t^2}}\,dt$ $= -2\sqrt{z} + 2\int \dfrac{1-t^2-1}{\sqrt{1-t^2}}\,dt$ $= -2\sqrt{1-t^2} + 2\int \sqrt{1-t^2} - \dfrac{1}{\sqrt{1-t^2}}\,dt$ $= -2\sqrt{1-t^2} + 2\left[\dfrac{t}{2}\sqrt{1-t^2} + \dfrac{1}{2}\sin^{-1}(t) - \sin^{-1}(t)\right] + c$ $= -2\sqrt{1-t^2} + 2\left[\dfrac{t}{2}\sqrt{1-t^2} - \dfrac{1}{2}\sin^{-1}t\right] + c$ $= -2\sqrt{1-t^2} + t\sqrt{1-t^2} - \sin^{-1}t + c$ replacing t by $\sqrt{x}$ $= I = -2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$ ans.
→	**Partial Fraction (Total : Types)** **Type : 1** all are linear factors (ax + b)
Q.2)	(a)$I = \int \dfrac{2x-1}{(x-1)(x+2)(x-3)}\,dx$ (b)$I = \int \dfrac{x^3}{(x-1)(x-2)}\,dx$ (c)$I = \int \dfrac{x^2}{(x-1)(x-2)(x-3)}\,dx$ (d)$I = \int \dfrac{3+4x-x^2}{(x+2)(x-1)}\,dx$
Sol.2)	(a)$I = \int \dfrac{2x-1}{(x-1)(x+2)(x-3)}\,dx$ let $\dfrac{2x-1}{(x-1)(x+2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$ $\Rightarrow 2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

$\Rightarrow 2x - 1 = A(x^2 - x - 6) + B(x^2 - 4x + 3) + C(x^2 + x - 2)$

Comp. the coefficients of x^2, x and constant term

$0 = A + B + C \quad \Rightarrow \quad C = -A - B$

$2 = -A - 4B + C \quad \Rightarrow \quad 2 = -2A - 5B$

$-1 = -6A + 3B - 2C \Rightarrow -1 = -4A - B$

solving these two equation we get

$A = \frac{-1}{6}, B = \frac{-1}{3}$ and $C = \frac{1}{2}$

$\therefore I = \int \frac{-1}{6(x-1)} - \frac{1}{3(x+2)} + \frac{1}{2(x-3)} dx$

$= \frac{-1}{6} \log |x - 1| - \frac{1}{3} \log |x + 2| + \frac{1}{2} \log |x - 3| + c$ \qquad ans.

(b) $I = \int \frac{x^3}{(x-1)(x-2)} dx$

Since degree of N^r > degree of D^r we have to divide

$\therefore I = \int (x + 3) + \frac{7x - 6}{(x-1)(x-2)} dx$

$= \frac{x^2}{2} + 3x + \int \frac{7x - 6}{(x-1)(x-2)} dx$

let $\frac{7x - 6}{(x-1)(x-2)} = \frac{A}{x-1} + Bx - 2$

$\Rightarrow 7x - 6 = A(x - 2) + B(x - 1)$

Comp. the coefficient of x and constant term

$7 = a + b$

$-6 = -2A - B$

$1 = -A$

$\therefore B = 8$

$A = -1$ and $B = 8$

$\therefore I = \frac{x^2}{2} + 3x + \int \frac{-1}{x-1} + \frac{8}{x-2} dx$

$I = \frac{x^2}{2} + 3x - \log |x - 1| + 8\log |x - 2| + c$ \qquad ans.

Q.3)	$I = \int \frac{x}{(x^2+1)(x^2+2)} dx$
Sol.3)	$I = \int \frac{x}{(x^2 + 1)(x^2 + 2)} dx$ put $x^2 = t$ $x\, dx = \frac{t}{2}$ $\therefore I = \frac{1}{2} \int \frac{dt}{(t+1)(t+2)}$ let $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$ Proceed Yourself $\frac{1}{2} [\log |x^2 + 1| - \log |x^2 + 2|] + c$ \qquad ans.
Q.4)	(a) $I = \int \frac{1}{\sin x - \sin(2x)} dx$

Sol.4)	(a) $I = \int \dfrac{1}{\sin x - \sin(2x)} dx$
	$= \int \dfrac{1}{\sin x \, 2\sin x \cos x} dx$
	$= \int \dfrac{1}{\sin x(1 - 2\cos x)} dx$
	multiply and divide by $\sin x$
	$= \int \dfrac{\sin x}{\sin^2 x(1 - 2\cos x)} dx$
	$= \int \dfrac{\sin x}{(1 - \sin^2 x)(1 - \cos x)} dx$
	$= \int \dfrac{\sin x}{(1 - \cos^2 x)(1 - 2\cos x)} dx$
	$= \int \dfrac{\sin x}{(1 - \cos x)(1 + \cos x)(1 - 2\cos x)} dx$
	put $\cos x = t$
	$\therefore \sin x \, dx = -dt$
	$\therefore I = -\int \dfrac{dt}{(1-t)(1+t)(1-2t)}$
	let $\dfrac{1}{(1-t)(1+t)(1-2t)} = \dfrac{A}{1-t} + \dfrac{B}{1+t} + \dfrac{C}{1-2t}$
	$\Rightarrow \quad 1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$
	$\Rightarrow \quad 1 = A(-2t^2 - t + 1) + B(2t^2 - 3t + 1) + C(1 - t^2)$
	Comp. the coefficient of t^2, t and constant term
	$0 = -2A + 2B - C \quad C = -2A + 2B$
	$0 = -A - 3B$
	$1 = A + B + C$
	$\therefore 1 = -A + 3B$
	$\quad 0 = -A - 3B$
	$\quad 1 = -2A$
	$\therefore A = \dfrac{-1}{2}, B = \dfrac{1}{6}$ and $C = \dfrac{4}{3}$
	$\therefore I = -\int \dfrac{-1}{2(1-t)} + \dfrac{1}{6(1+t)} + \dfrac{4}{3(1-2t)} dt$
	$= \left[\dfrac{+1}{2}\log\mid 1-t \mid + \dfrac{1}{6}\log\mid 1+t \mid + \dfrac{4}{3}\log\mid 1-2t\mid.\left(\dfrac{-1}{2}\right)\right] + c$
	$= \dfrac{-1}{2}\log\mid 1-t \mid - \dfrac{1}{6}\log\mid 1+t \mid + \dfrac{2}{3}\log\mid 1-2t \mid + c$
	replacing t
	$= I = \dfrac{-1}{2}\mid 1-\cos x \mid - \dfrac{1}{6}\log\mid 1+\cos x \mid + \dfrac{2}{3}\log\mid 1-\cos x \mid + c \qquad$ ans.
→	**Type : 2** Linear and Quadratic Faction
Q.5)	(a) $I = \int \dfrac{x}{(x-1)(x^2+4)} dx$ $\qquad\qquad$ (b) $I = \int \dfrac{1}{1+x+x^2+x^3} dx$
Sol.5)	(a) $I = \int \dfrac{x}{(x-1)(x^2+4)} dx$
	let $\dfrac{x}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4} dx$
	$\Rightarrow \quad x = A(x^2 + 4) + (Bx + c)(x - 1)$
	$\Rightarrow \quad x = A(x^2 + 4) + (Bx^2 - Bx + Cx - C)$

Comp. the coefficient of x^2 , x and constant term

$0 = A + B$

$1 = -B + C$

$0 = 4A - C$

Solving these equations,

we get $A = \frac{1}{5} , B = \frac{-1}{5}$ and $C = \frac{4}{5}$

$\therefore I = \int \frac{1}{5(x-1)} + \frac{\frac{-1}{5}x+\frac{4}{5}}{x^2+4} dx$

$= \frac{1}{5}\int \frac{1}{x-1} dx - \frac{1}{5}\int \frac{x}{x^2+4} dx + \frac{4}{5}\int \frac{1}{x^2+4} dx$

put $x^2 + 4 = t$

$\therefore x\, dx = \frac{dt}{2}$

$\therefore I = \frac{1}{5}\log|x-1| - \frac{1}{10}\int \frac{dt}{t} + \frac{4}{5} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$I = \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + c$ ans.

(b) $I = \int \frac{1}{1+x+x^2+x^3} dx$

$= \int \frac{1}{(1+x)+x^2(1+x)} dx$

$= \int \frac{1}{(1+x)(1+x^2)} dx$

let $\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+c}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)(x+1)$

$1 = A(x^2+1) + (Bx^2+Bx+Cx+C)$

Comp. the coefficient of x^2 , x and constant

$0 = A + B \Rightarrow B = -A$

$0 = B + C \quad \therefore 0 = -A + C$

$1 = A + C \quad 1 = A + C$

$1 = 2C$

$C = \frac{1}{2}, A = \frac{1}{2}$ and $B = \frac{-1}{2}$

$\therefore I = \frac{1}{2(x+1)} + \frac{\frac{-1}{2}+\frac{1}{2}}{x^2+1} dx$

$= \frac{1}{2}\int \frac{1}{x+1} dx - \frac{1}{2}\int \frac{x}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2+1} dx$

put $x^2 + 1 = t$

$dx = \frac{dt}{2}$

$= \frac{1}{2}\log|x+1| - \frac{1}{4}\int \frac{dt}{t} + \frac{1}{2}\tan^{-1}x$

$I = \frac{-1}{2}\log|x+1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + c$ ans.

Q.6)	$I = \int \dfrac{x}{x^3-1} dx$
Sol.6)	$I = \int \dfrac{x}{x^3-1} dx$

$$= \int \frac{x}{(1-x)(x^2+x+1)} dx$$

let $\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+c}{x^2+x+1}$

$x = A(x^2 + x + 1) + (Bx + c)(x - 1)$

$x = A(x^2 + x + 1) + (Bx - Bx^2 + cx + c)$

Comp. the coefficient of x^2, x and constant term

$0 = A + B \Rightarrow B = -A$

$1 = A - B + C \Rightarrow 1 = 2A + C$

$0 = A - C \Rightarrow 0 = A - C$

$1 = 3A$

$A = \frac{1}{3}, B = \frac{-1}{3}$ and $C = \frac{1}{3}$

$$\therefore I = \int \frac{1}{3(x-1)} + \frac{\frac{-1}{3}x + \frac{1}{3}}{x^2+x+1} dx$$

$$= \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{x-1}{x^2+x+1} dx \quad \left\{ type \int \frac{linear}{quadratic} \right\}$$

$$= I = \frac{1}{3} \log |x-1| - \frac{1}{3} I \quad ....(1)$$

where $I = \int \frac{x-1}{x^2+x+1} dx$

Proceed Yourself and get I

$$I = \frac{1}{3} \log |x-1| - \frac{1}{6} \log |x^2 + x + 1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$$

Q.7)	(a)$I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta} d\theta$
Sol.7)	(a)$I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta} d\theta$ $= \int \frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta} d\theta$ $= \int \frac{\tan\theta . sec^2\theta}{1+\tan^3\theta} d\theta$ put $\tan\theta = t$ $\quad sec^2\theta \, d\theta = dt$ $\therefore I = \int \frac{t}{1+t^3} dt$ $= \int \frac{t}{(1+t)(t^2-t+1)} dt$ Proceed as above Qns. :- $\frac{-1}{3} \log |1 + \tan\theta| + \frac{1}{6} \log |\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right)$ ans.
→	**Type : 3 Linear and Linear repeating factors**
Q.8)	(a)$I = \int \frac{3x+1}{(x+2)(x+2)(x-2)^2} dx$ (b)$I = \int \frac{x^2+x+1}{(x-1)^3} dx$
Sol.8)	(a)$I = \int \frac{3x+1}{(x+2)(x+2)(x-2)^2} dx$ let $\frac{3x+1}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ $3x + 1 = A(x-2)^2 + B(x-2)(x+2) + c(x+2)$ $3x + 1 = A(x^2 - 4x + 4) + B(x^2 - 4) + C(x+2)$

Comp. the coefficient of x^2, x and constant

$0 = A + B \quad \Rightarrow \quad B = -A$

$3 = -4A + C \quad \Rightarrow \quad 3 = -4A + c$

$1 = 4A - 4B + 2C \Rightarrow 1 = 8A + 2C$

solving these equations, we get

$A = \frac{-5}{16}, B = \frac{5}{16}$ and $C = \frac{7}{4}$

$\therefore I = \frac{-5}{16(x+2)} + \frac{5}{16(x-2)} + \frac{7}{4(x-2)^2} dx$

$I = \frac{-5}{16} \log |x+2| + \frac{5}{16} \log |x-2| - \frac{7}{4(x-2)} + c$ ans. .....$\left\{ \text{Since} \int \frac{1}{x^2} dx = \frac{-1}{x} \right\}$

(b)$I = \int \frac{x^2+x+1}{(x-1)^3} dx$

let $\frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$

$x^2 + x + 1 = A(x-1)^2 + B(x-1) + C$

$x^2 + x + 1 = A(x^2 - 2x + 1) + B(x-1) + C$

Comp. the coefficient of x^2 , x and constant

$1 = A$

$1 = -2A + B$

$1 = A - B + C$

solving these equation we get $A = 1, B = 3, C = 3$

$\therefore I = \int \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3} dx$

$= \log |x-1| - \frac{3}{(x-1)} + 3 \int (x-1)^{-3} dx$

$= \log |x-1| - \frac{3}{x-1} + 3 \frac{(x-1)^{-2}}{-2} + c$

$\therefore I = \log |x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + c$ ans.

Q.9)	$I = \int \frac{3x+5}{x^3-x^2-x+1} dx$
Sol.9)	$I = \int \frac{3x+5}{x^3-x^2-x+1} dx$ $= \int \frac{3x+5}{x^2(x-1)-1(x-1)} dx$ $= \int \frac{3x+5}{(x-1)(x^2-1)} dx$ $= \int \frac{3x+5}{(x-1)(x+1)(x-1)} dx$ $= \int \frac{3x+5}{(x+1)(x-1)^2} dx$ let $\frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ $3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$ $3x + 5 = A(x^2 - 2x + 1) + B(x^2 - 1) + C(x+1)$ $0 = A + B$ $3 = -2A + C$ $5 = A - B + C$

	solving these equation , we get $A = \frac{1}{2}, B = \frac{-1}{2}, C =$
	$\therefore I = \int \frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2} dx$
	$I = \frac{1}{2}\log\mid x+1 \mid -\frac{1}{2}\log \mid x-1 \mid -\frac{4}{x-1} + c$ ans.
→	**Type : 4** <u>Even Power of x let $x^2 = y$ (temp.)</u>
Q.10)	(a)$I = \int \frac{x^2}{(x^2+1)(x+4)} dx$ (b)$I = \int \frac{1}{(x^4-1)} dx$
Sol.10)	(a)$I = \int \frac{x^2}{(x^2+1)(x^{+4})} dx$
	let $x^2 = y$
	$\therefore \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$
	let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$
	$y = A(y+4) + B(y+1)$
	Comp. coefficient of y and constant
	$1 = A + B$
	$0 = 4A + B$
	$1 = -3A$
	$A = \frac{-1}{3} \therefore B = \frac{4}{3}$
	$\therefore I = \int \frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)} dx$
	$\quad = \frac{-1}{3}\int \frac{1}{(x^2+1)} dx + \frac{4}{3}\int \frac{1}{x^2+(2)^2} dx$
	$\quad = \frac{-1}{3}\tan^{-1}x + \frac{4}{3} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ ans.
	(b)$I = \int \frac{1}{(x^4-1)} dx$
	$\quad = \int \frac{1}{(x^2+1)(x^2-1)} dx$
	let $x^2 = y$
	$\therefore \frac{1}{(x^2+1)(x^2-1)} = \frac{1}{(y+1)(y-1)}$
	let $\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$
	$1 = A(y-1) + B(y+1)$
	Comp.
	$0 = A + B$
	$1 = -A + B$
	$1 = 2B$
	$\therefore B = \frac{1}{2}$ and $A = \frac{-1}{2}$
	$\therefore I = \int \frac{-1}{2(x^2+1)} + \frac{1}{2(x^2-1)} dx$
	$\quad = \frac{-1}{2}\int \frac{1}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2-1} dx$

$$= \frac{-1}{2}\tan^{-1}x + \frac{1}{2} \times \frac{1}{2 \times 1}\log\left|\frac{x-1}{x+1}\right| + c$$

$$= \frac{-1}{2}\tan^{-1}x + \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + c \qquad \text{ans.}$$