CHAPTER – 7
INTEGRALS

I. Integrals of Type : $\int \sqrt{ax^2 + bx + c}\, dx$

In order to evaluate the above type of integral, we put $ax^2 + bx + c$ in the form :

$$\begin{cases} a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a}\right)^2\right] & \text{when } b^2 < 4ac. \\ a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] & \text{when } b^2 > 4ac. \end{cases}$$

and the integral reduces to one of the following three forms:

$\int \sqrt{a^2 + x^2}\,dx$, $\int \sqrt{a^2 - x^2}\,dx$, $\int \sqrt{x^2 - a^2}\,dx$, which can be evaluated by using standard formulae.

Example 1: Evaluate

(i) $\int \sqrt{x^2 + 4x + 8}\, dx$, (ii) $\int \sqrt{(x - 5)(7 - x)}\, dx$

(ii) $\int \sqrt{14x - 20 - 2x^2}\, dx$ (iv) $\int \sqrt{4a - x^2}\, dx$

Solution : (i) $\int \sqrt{x^2 + 4x + 8}\, dx = \int \sqrt{x^2 + 4x + 4 + 4}\, dx$

$$= \int \sqrt{(x+2)^2 + (2)^2}\, dx$$

$$= \frac{x+2}{2}\sqrt{x^2 + 4x + 8} + \frac{4}{2}\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

$$= \frac{1}{2}(x+2)\sqrt{x^2 + 4x + 8} + 2\log\left|(x+2) + \sqrt{x^2 + 4x + 8}\right| + c$$

(ii) $\int \sqrt{(x - 5)(7 - x)}\, dx = \int \sqrt{12x - 35 - x^2}\, dx$

$$= \int \sqrt{-35 - (x^2 - 12x)}\,dx = \int \sqrt{-35 - (x^2 - 12x + 36 - 36)}\, dx$$

$$= \int \sqrt{(36 - 35) - (x - 6)^2}\,dx = \int \sqrt{1^2 - (x - 6)^2}\, dx$$

$$= \frac{x-6}{2}\sqrt{(x-5)(7-x)} + \frac{1}{2}\sin^{-1}\left(\frac{x-6}{1}\right) + c$$

(iii) $\int \sqrt{14x - 20 - 2x^2}\, dx = \sqrt{2}\int \sqrt{-10 + 7x - x^2}\, dx$

$$= \sqrt{2}\int\sqrt{-10 - (x^2 - 7x)}\, dx = \sqrt{2}\int\sqrt{-10 - \left(x^2 - 7x + \frac{49}{4} - \frac{49}{4}\right)}\, dx$$

$$= \sqrt{2}\int\sqrt{-10 + \frac{49}{4} - \left(x - \frac{7}{2}\right)^2}\, dx = \sqrt{2}\int\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\, dx$$

$$= \sqrt{2}\left[\frac{x - 7/2}{2}\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - 7/2\right)^2} + \frac{9}{8}\sin^{-1}\left(\frac{x - \frac{7}{2}}{\frac{3}{2}}\right)\right] + c$$

$$= \sqrt{2}\left[\frac{2x-7}{4}\sqrt{7x - 10 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right)\right] + c$$

(iv) $\int\sqrt{4ax - x^2}\, dx = \int\sqrt{-\left(x^2 - 4ax + 4a^2 - 4a^2\right)}\, dx$

$$= \int\sqrt{4a^2 - \left(x^2 - 4ax + 4a^2\right)}\, dx = \int\sqrt{(2a)^2 - (x - 2a)^2}\, dx$$

$$= \frac{x-2a}{2}\sqrt{4ax - x^2} + \frac{4a^2}{2}\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

$$= \frac{1}{2}(x-2a)\sqrt{4ax - x^2} + 2a^2\sin^{-1}\left(\frac{x-2a}{2a}\right) + c$$

II. Integrals of Type : $\int (px+q)\sqrt{ax^2+bx+c}\, dx$

Here, px + q is written as

$px + q = A\left[\frac{d}{dx}\left(ax^2+bx+c\right)\right] + B$ and the values of A and B are determined by equating the coefficients of x and constant terms on both sides.

Then, writing

$$\int (px+q)\sqrt{ax^2+bx+c}\, dx = A\int (2ax+b)\sqrt{ax^2+bx+c}\, dx + B\int\sqrt{ax^2+bx+c}\, dx$$

$$= \frac{2A}{3}\,(ax^2 + bx + c)^{3/2} + B\int \sqrt{ax^2 + bx + c}\ dx$$

The second part is evaluated as explained in above example.

Example 2: Evaluate

(i) $\int (x-3)\sqrt{x^2+4x+3}\ dx$ (ii) $\int (3x+5)\sqrt{2x^2+3x+7}\ dx$

(ii) $\int (x-4)\sqrt{4+3x-x^2}\ dx$ (iii) $\int (5x-1)\sqrt{6+5x-2x^2}\ dx$

Solution :

(i) Let $I = \int (x-3)\sqrt{x^2+4x+3}\ dx$

$$x - 3 = A(2x+4) + B$$

$$\Rightarrow A = \frac{1}{2},\ B = -5$$

$$\therefore I = \int \frac{1}{2}(2x+4)\sqrt{x^2+4x+3}\ dx - 5\int \sqrt{x^2+4x+3}\ dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\int \sqrt{(x+2)^2 - (1)^2}\ dx$$

$$= \frac{1}{3}\left(x^2+4x+3\right)^{3/2} - 5\left[\frac{x+2}{2}\sqrt{x^2+4x+3}\ \frac{-1}{2}\log\left|(x+2) + \sqrt{x^2+4x+3}\right|\right] + c$$

(ii) $I = \int (3x+5)\sqrt{2x^2+3x+7}\ dx$

$$3x + 5 = A(4x+3) + B \Rightarrow 4A = 3 \text{ and } 3A + B = 5$$

$$\Rightarrow A = 3/4 \text{ and } B = \frac{11}{4}$$

$$I = \int \frac{3}{4}(4x+3)\sqrt{2x^2+3x+7}\ dx + \frac{11}{4}\int \sqrt{2}\sqrt{x^2 + \frac{3}{2}x + \frac{7}{2}}\ dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\int \sqrt{\left(x + 3/4\right)^2 + \frac{7}{2} - \frac{9}{16}}\ dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\int\sqrt{\left(x+\tfrac{3}{4}\right)^2+\left(\frac{\sqrt{47}}{4}\right)^2}\,dx$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\left[\frac{x+\frac{3}{4}}{2}\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}+\frac{47}{8}\log\left|\left(x+\tfrac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}\right|\right]+C$$

$$= \frac{1}{2}\left(2x^2+3x+7\right)^{3/2} + \frac{11\sqrt{2}}{4}\left[\frac{4x+3}{8}+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}+\frac{47}{8}\log\left|\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+\frac{7}{2}}\right|\right]+C$$

(iii) $I=\int(x-4)\sqrt{4+3x-x^2}\,dx$

$$x-4=A(3-2x)+B \Rightarrow A=-\frac{1}{2},\ B=-\frac{5}{2}$$

$$\therefore I=\int -\frac{1}{2}(3-2x)\sqrt{4+3x-x^2}\,dx+\left(\frac{-5}{2}\right)\int\sqrt{4+3x-x^2}\,dx$$

$$= -\frac{1}{3}\left(4-3x-x^2\right)^{3/2}-\frac{5}{2}\left[\frac{2x-3}{4}\sqrt{4+3x-x^2}+\frac{25}{8}\sin^{-1}\frac{2x-3}{5}\right]+C$$

(iv) $\therefore I=\int(5x-1)\sqrt{6+5x-2x^2}\,dx$

$$5x-1=A[5-4x]B,\ A=-\tfrac{5}{4},\ B=\frac{21}{4}$$

$$\therefore I=\int\frac{-5}{4}(5-4x)\sqrt{6+5x-2x^2}\,dx+\left(\frac{21}{4}\right)\int\sqrt{2}\sqrt{3+\frac{5}{2}x-x^2}\,dx$$

$$= \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2}+\frac{21\sqrt{2}}{4}\int\sqrt{\left(\frac{73}{4}\right)^2-\left(x-\tfrac{5}{4}\right)^2}\,dx$$

$$= \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2}+\frac{21\sqrt{2}}{4}\left[\frac{x-\frac{5}{4}}{2}\sqrt{3+\frac{5}{2}x-x^2}+\frac{73}{32}\sin^{-1}\left(\frac{4x-5}{\sqrt{73}}\right)\right]+C$$

$$= \frac{-5}{6}\left(6+5x-2x^2\right)^{3/2}+\frac{21\sqrt{2}}{4}\left[\frac{4x-5}{8}\sqrt{3+\frac{5}{2}x-x^2}+\frac{73}{32}\sin^{-1}\left(\frac{4x-5}{\sqrt{73}}\right)\right]+C$$

Note: The above integral can also be evalnated by substituting $\tan \frac{x}{2} = t$.

Exercise 1.

1. Evaluate:

(i) $\int \sqrt{3x^2 + 4x + 1}\, dx$ (ii) $\int \sqrt{1 + 2x - 3x^2}\, dx$

(iii) $\int \sqrt{x^2 + 4x + 1}\, dx$ (iv) $\int \sqrt{3 - 2x - 2x^2}\, dx$

(v) $\int \sqrt{(x-3)(5-x)}\, dx$ (vi) $\int \sqrt{(2ax - x^2)}\, dx$

2. Evaluate:

(i) $\int (2x+3)\sqrt{x^2 + 4x + 3}\, dx$ (ii) $\int (2x-5)\sqrt{2 + 3x - x^2}\, dx$

(iii) $\int (2x+3)\sqrt{4x^2 + 5x + 6}\, dx$ (iv) $\int (2x-5)\sqrt{x^2 + 4x + 3}\, dx$

(v) $\int (x+1)\sqrt{1 - x - x^2}\, dx$ (vi) $\int (6x+5)\sqrt{x + 6 - 2x^2}\, dx$

ANSWERS:

EXERCISE : 1

1.(i) $\frac{1}{6}(3x+2)\sqrt{3x^2 + 4x + 1} - \frac{\sqrt{3}}{18} \log \left| \left(x + \frac{2}{3}\right) + \sqrt{x^2 + \frac{4}{3}x + \frac{1}{3}} \right| + C$

(ii) $\frac{1}{6}(3x-1)\sqrt{1 + 2x - 3x^2} + \frac{2\sqrt{3}}{9} \sin^{-1}\left(\frac{3x-1}{2}\right) + C$

(iii) $\frac{1}{2}(x+2)\sqrt{x^2 + 4x + 1} - \frac{3}{2} \log \left| (x+2) + \sqrt{x^2 + 4x + 1} \right| + C$

(iv) $\sqrt{2}\left(\frac{2x+1}{4}\right)\sqrt{\frac{3}{2} - x - x^2} + \frac{7}{8} \sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C$

(v) $\frac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \frac{1}{2} \sin^{-1}(x-4) + C$

(vi) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + C$

Q.2(i) $\frac{2}{3}\left(x^2+4x+3\right)^{3/2} - \frac{x+2}{2}\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|\,(x+2)+\sqrt{x^2+4x+3}\right|+C$

(ii) $\frac{-2}{3}\left(2+3x-x^2\right)^{3/2} - \left(\frac{2x-3}{2}\right)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\frac{(2x-3)}{\sqrt{17}}+C$

(iii) $\frac{1}{6}\left(4x^2+5x+6\right)^{3/2} + \frac{7}{16}(8x+5)\sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}+\frac{497}{256}\log\left|\frac{(8x+5)}{8}+\sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}\right|+C$

(iv) $\frac{2}{3}\left(x^2-4x+3\right)^{3/2} - \frac{1}{2}(x-2)\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right|+C$

(v) $\frac{-1}{3}\left(1-x-x^2\right)^{3/2} + \frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$

(vi) $-\left(6+x-2x^2\right)^{3/2} + \frac{13}{16}(4x-1)\sqrt{6+x-2x^2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$