DEFINITE INTEGRALS

Q.1)	$I = \int_3^4 e^{3x} + 2x\, dx$
Sol.1)	$a = 3, b = 4, nh = 4 - 3 = 1$ $I = \lim_{h\to0} h[f(3) + f(3+h) + f(3+2h) + \ldots\ldots f(3+(n-1)h)]$ $= \lim_{h\to 0} h[(e^9+6) + (e^{9+3h} + 6 + 2h) + (e^{9+6h} + 6 + 4h) + \ldots\ldots (e^{9+3(n-1)h} + 6 + 2(n-1)h)]$ $= \lim_{h\to 0} h[\{e^9 + e^{9+3h} + e^{9+6h} + \ldots\ldots e^{9+3(n-1)h}\} + \{(6+6+6+\cdots\ldots n\ term) + (2h + 4h + \ldots\ldots 2(n-1)h)\}]$ $= \lim_{h\to 0} h[e^9 + e^{9+3h} + e^{9+6h} + \ldots\ldots e^{9+3(n-1)h}] + \lim_{h\to0} h[6n + 2h(1 + 2 + \cdots.(n-1)]$ $= e^9 \lim_{h\to 0} h[1 + e^{3h} + e^{6h} + \ldots\ldots e^{3(n-1)h}] + \lim_{h\to0} h\left[6n + \frac{2h.n(n-1)}{2}\right]$ $= e^9 \lim_{h\to 0} h\left[1\left(\frac{(e^{3h})^n - 1}{e^{3h}-1}\right)\right] + \lim_{h\to0}[6nh + (nh)(nh-h)]$ $= e^9 \lim_{h\to 0} h\left[\frac{e^{3nh}-1}{\frac{e^{3nh}-1}{3h}\times 3h}\right] + \lim_{h\to0}[6n + (nh)(nh-h)]$ Put $nh = 1$ $\therefore I = e^9 \frac{(e^3-1)}{\lim_{h\to0}\left(\frac{e^{3h}-1}{3h}\right)\times 3} + \lim_{h\to0}[6 + (1)(1-h)]$ $I = \frac{e^9(e^3-1)}{3} + 6 + 1$ $I = \frac{e^9(e^3-1)}{3} + 7$ Ans.....
Q.2)	Evaluate $\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$
Sol.2)	$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2} cosec^2\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right)\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \cot\left(\frac{x}{2}\right) dx + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^x . cosec^2\left(\frac{x}{2}\right) dx$ $I = -\left[\left(\cot\frac{x}{2}.e^x\right)_{\pi/2}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{2}.cosec^2\left(\frac{x}{2}\right) e^x\, dx\right] + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^x cosec^2\frac{x}{2}$ $I = -\left[\cot\frac{\pi}{2}.e^x - \cot\frac{\pi}{4}.e^{\frac{\pi}{2}}\right] - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^x cosec^2\frac{x}{2} dx + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} e^x cosec^2\left(\frac{x}{2}\right) dx$ $I = -\left[0 - e^{\frac{\pi}{2}}\right]$ ……..$\left\{\because \cot\left(\frac{\pi}{2}\right) = 0, \cot\left(\frac{\pi}{4}\right) = 1\right\}$

	$\therefore\ I = e^{\frac{\pi}{2}}$ *Ans .....*
Q.3)	$I = \int_0^{\frac{\pi}{2}} sin(2x).tan^{-1}(sin\,x)\,dx$
Sol.3)	$I = 2\int_0^{\frac{\pi}{2}} sin\,x.cos\,x.tan^{-1}(sin\,x)\,dx$ Put $sin\,x = t$ when $x = 0\ ;\ t = 0$ $\cos x\,dx = dt$ when $x = \frac{\pi}{2}\ ;\ t = 1$ $\therefore\ I = 2\int_0^1 t\,.tan^{-1}(t)\,dt$ $= 2\left[\left(tan^{-1}t.\frac{t^2}{2}\right)_0^1 - \int_0^1 \frac{1}{1+t^2}.\frac{t^2}{2}\,dt\right]$ $= 2\left[\left(\frac{\pi}{4}.\frac{1}{2}\right) - 0 - \frac{1}{2}\int_0^1 \frac{t^2}{1+t^2}\,dt\right]$ $= 2\left[\frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{1+t^2-1}{1+t^2}dt\right]$ $= \frac{\pi}{4} - \int_0^1 1 - \frac{1}{1+t^2}dt$ $= \frac{\pi}{4} - [t - tan^{-1}t]_0^1$ $= \frac{\pi}{4} - \left[\left(1 - \frac{\pi}{4}\right) - 0\right]$ $I = \frac{\pi}{2} - 1$ Ans....
Q.4)	$I = \int_0^{\frac{\pi}{2}} \frac{x+sin\,x}{1+cos\,x}dx$
Sol.4)	$I = \int_0^{\frac{\pi}{2}} \frac{x+2\,sin\left(\frac{x}{2}\right).\,cos\left(\frac{x}{2}\right)}{2\,cos^2\left(\frac{x}{2}\right)}dx$ $I = \int_0^{\frac{\pi}{2}} \frac{x}{2cos^2(x/2)} + \frac{2\,sin\left(\frac{x}{2}\right).\,cos\left(\frac{x}{2}\right)}{2\,cos^2\left(\frac{x}{2}\right)}dx$ $I = \int_0^{\frac{\pi}{2}} \frac{1}{2}x\,sec^2\left(\frac{x}{2}\right) + tan\left(\frac{x}{2}\right)dx$ $I = \frac{1}{2}\int_0^{\frac{\pi}{2}} x\,.sec^2\left(\frac{x}{2}\right)dx + \int_0^{\frac{\pi}{2}} tan\left(\frac{x}{2}\right)dx$ $I = \frac{1}{2}\left[\left(x\,.tan\left(\frac{x}{2}\right).2\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1\,.tan\left(\frac{x}{2}\right).2\,dx\right] + \int_0^{\frac{\pi}{2}} tan\left(\frac{x}{2}\right)dx$ $I = \frac{1}{2}\left[\left(\frac{\pi}{2}.tan\frac{\pi}{4}.2\right) - 0\right] - \int_0^{\frac{\pi}{2}} tan\frac{x}{2}dx + \int_0^{\frac{\pi}{2}} tan\frac{x}{2}dx$ $I = \frac{1}{2}\left[\frac{\pi}{1}\right] = \frac{\pi}{2}$ *Ans*
Q.5)	$I = \int_0^{\frac{\pi}{4}} sin^3(2t)\,.cos(2t)\,dt$
Sol.5)	(a) I = $\int_0^{\frac{\pi}{4}} sin^3(2t)\,.cos(2t)\,dt$ Put $sin\,(2t) = z$ $\therefore\ 2\cos(2t)dt = dz$ $when\ t = 0\ ;\ z = 0$

	$\cos(2t)\,dt = \frac{dz}{2}$ when $t = \frac{\pi}{4}$; $z = 1$ $\therefore I = \frac{1}{2}\int_0^1 z^3\,dz$ $= \frac{1}{2}\left(\frac{z^4}{4}\right)_0^1$ $= \frac{1}{2}\left(\frac{1}{4} - 0\right)$ $I = \frac{1}{8}$ Ans.....
Q.6)	$I = \int_4^9 \frac{\sqrt{x}}{\left(30 - e^{\frac{3}{2}}\right)^2}\,dx$
Sol.6)	$I = \int_4^9 \frac{\sqrt{x}}{\left(30 - e^{\frac{3}{2}}\right)^2}\,dx$ Put $30 - x^{3/2} = t$ when $x = 4$; $t = 30 - 8 = 22$ $\frac{-3}{2}\,x^{1/2}\,dx = dt$ when $x = 9$; $t = 30 - 27 = 3$ $\sqrt{x}dx = \frac{-2}{3}dt$ $\therefore I = \frac{-2}{3}\int_{22}^{3}\frac{dt}{t^2}$ $= \frac{-2}{3}\left(-\frac{1}{t}\right)_{22}^{3}$ $= \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right]$ $= \frac{2}{3}\left(\frac{22-3}{66}\right) = \frac{2(19)}{3\times66} = \frac{19}{99}$ Ans....
Q.7)	$\int_0^{\frac{\pi}{2}} \sqrt{\sin\varphi}\,\cos^5\varphi\,d\varphi$
Sol.7)	$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\varphi}\,.\cos^4\varphi.\cos\varphi\,d\varphi$ Put $\sin\varphi = t$ when $\varphi = 0$; $t = 0$ $\cos\varphi\,d\varphi = dt$ when $\varphi = \frac{\pi}{2}$; $t = 1$ $\therefore I = \int_0^1 \sqrt{t}(1 - \sin^2\varphi)^2\,.dt$ $= \int_0^1 \sqrt{t}(1 - t^2)^2\,dt$ $= \int_0^1 \sqrt{t}(1 + t^4 - 2t^2)\,dt$ $= \int_0^1 \sqrt{t} + t^{9/2} - 2t^{5/2}\,dt$ $= \left(\frac{2}{3}t^{3/2} + \frac{2}{11}t^{11/2} - 2\,.\frac{2}{7}t^{7/2}\right)_0^1$ $= \left(\frac{2}{3} + \frac{2}{11} - \frac{4}{7}\right) - (0)$ $= \frac{154+42-132}{231}$

	$= \frac{64}{231}$ *Ans* ...
Q.8)	$I = \int_0^1 sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$
Sol.8)	$I = \int_0^1 sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ $I = 2\int_0^1 tan^{-1}x\, dx$ ....$\left\{\because\ 2tan^{-1}x = sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}$ $I = 2\int_0^1 tan^{-1}x\,.1\,dx$ $= 2\left[(tan^{-1}x\,.x)_0^1 - \int_0^1 \frac{1}{1+x^2}.x\,dx\right]$ Put $1 + x^2 = t$ $when\ x = 0\ ;\ t = 1$ $x\,dx = \frac{dt}{2}$ $x = 1\ ,\ t = 2$ $\therefore I = 2\left[\left(\frac{\pi}{4}.1\right) - (0) - \frac{1}{2}\right]_1^2 \frac{dt}{t}$ $= 2\left[\frac{\pi}{4} - \frac{1}{2}(\log t)_1^2\right]$ $= 2\left[\frac{\pi}{4} - \frac{1}{2}(log\ 2 - log\ 1)\right]$ $= 2\left[\frac{\pi}{4} - \frac{1}{2}log\ 2\right]$ ... ...$\{\because\ log\ 1 = 0\}$ $I = \frac{\pi}{2} - log\ 2$ *Ans* ...
Q.9)	$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} dx$
Sol.9)	$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} dx$ *Rationalize* $I = \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$ $I = \left[\frac{2}{3}(1+x)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^1$ $= \frac{2}{3}\left[(2)^{\frac{3}{2}} + (1)^{\frac{3}{2}}\right] - \frac{2}{3}\left[(1)^{\frac{3}{2}} + 0\right]$ $= \frac{2}{3}(2\sqrt{2}+1) - \frac{2}{3}(1)$ $= \frac{4\sqrt{3}}{3} + \frac{2}{3} - \frac{2}{3}$ $I = \frac{4\sqrt{2}}{3}$ ans.
Q.10)	$I = \int_0^1 \frac{2x+3}{5x^2+1} dx$
Sol.10)	$I = \int_0^1 \frac{2x+3}{5x^2+1} dx$ $(separate)$ $I = 2\int_0^1 \frac{x}{5x^2+1} dx + 3\int_0^1 \frac{1}{5x^2+1} dx$ Put $5x^2 + 1 = t$ $when\ x = 0\ ;\ t = 1$ $10x\,dx = dt$ $when\ x = 1\ ;\ t = 6$

	$x\,dx = \frac{dt}{10}$ $\therefore\ I = \frac{2}{10}\int_1^6 \frac{dt}{t} + \frac{3}{5}\int_0^1 \frac{1}{x^2+\left(\frac{1}{\sqrt{5}}\right)^2}dt$ $= \frac{1}{5}[log\ t]_1^6 + \frac{3}{5}\times\sqrt{5}\left[tan^{-1}(x\sqrt{5})\right]_0^1$ $= \frac{1}{5}\left[(\log 6 - \log 1) + \frac{3}{\sqrt{5}}\left(tan^{-1}\sqrt{5} - tan^{-1}0\right)\right]$ $I = \frac{1}{5}log\ 6 + \frac{3}{\sqrt{5}}tan^{-1}(\sqrt{5})$ ans.