



DEFINITE INTEGRALS

Q.1)	$\int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$
Sol.1)	$I = \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx \quad \dots\dots(1)$ $I = \int_0^1 \tan^{-1} \left(\frac{2(1-x)-1}{1+(1-x)-(1-x)^2} \right) dx \quad \dots\dots(P-IV)$ $I = \int_0^1 \tan^{-1} \left(\frac{2-2x-1}{1+1-x-1-x^2+2x} \right) dx$ $I = \int_0^1 \tan^{-1} \left(\frac{-2x+1}{1+x-x^2} \right) dx$ $I = - \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx \quad \dots\dots(2) \quad \dots\dots\{\because \tan^{-1}(-x) = -\tan^{-1}x\}$ $(1) + (2)$ $2I = 0$ $I = 0 \quad ans.$ <p>Alternate:</p> $I = \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$ $I = \int_0^1 \tan^{-1} \left(\frac{(x)+(x-1)}{1-x(x-1)} \right) dx \quad \dots\{adjustment\}$ $I = \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(x-1) dx \quad \dots\dots \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1}x + \tan^{-1}y \right]$ $I = \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}[(1-x)-1] dx \quad \dots\dots (P-IV)$ $I = \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(-x) dx$ $I = \int_0^1 \tan^{-1}x dx - \int_0^1 \tan^{-1}x dx \quad \dots\dots [\tan^{-1}(-x) = -\tan^{-1}x]$ $I = 0 \quad ans.$
Q.2)	$\int_0^1 \cot^{-1}(1-x+x^2) dx$
Sol.2)	$I = \int_0^1 \cot^{-1}(1-x+x^2) dx$ $I = \int_0^1 \tan^{-1} \left(\frac{1}{1-x+x^2} \right) dx \quad \dots\dots \left[\tan^{-1} \left(\frac{1}{x} \right) = \cot^{-1}x \right] I =$ $\int_0^1 \tan^{-1} \left[\frac{x+(1-x)}{1-x(1-x)} \right] dx \quad \dots\dots [adjustment]$ $I = \int_0^1 \tan^{-1}(x) dx + \int_0^1 \tan^{-1}(1-x) dx \quad \dots \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1}x + \tan^{-1}y \right]$ $I = \int_0^1 \tan^{-1}(x) dx + \int_0^1 \tan^{-1}[1-(1-x)] dx \quad \dots\dots (P-IV)$ $I = \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}(x) dx$ $I = 2 \int_0^1 \tan^{-1}x dx$ $I = 2 \int_0^1 \tan^{-1}x .1 dx$



$$\begin{aligned}
 I &= 2 \left[(\tan^{-1}x - x) \Big|_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x \, dx \right] \\
 I &= 2 \left[\left(\frac{\pi}{4} - 0 \right) - \int_0^1 \frac{x}{1+x^2} \, dx \right] \\
 \text{Put } 1+x^2 &= t \quad \text{when } x = 0 ; \\
 x \, dx &= \frac{dt}{2} \quad \text{when } x = 1 ; t = 2 \\
 \therefore I &= 2 \left[\frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{dt}{t} \right] \\
 &= \frac{\pi}{2} - \int_1^2 \frac{dt}{t} \\
 &= \frac{\pi}{2} - [\log t]_1^2 \\
 &= \frac{\pi}{2} - [\log 2 - \log 1]_1^2 \\
 I &= \frac{\pi}{2} - \log 2 \quad \text{ans.} \quad [\because \log(1) = 0]
 \end{aligned}$$

Q.3) $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} \, dx \dots \dots \dots \text{[Removal of } x\text{]}$

$$\begin{aligned}
 \text{Sol.3)} \quad I &= \int_0^\pi \frac{x \sin x}{1+\cos^2 x} \, dx \quad \dots \dots \dots (1) \\
 I &= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} \, dx \quad \dots \dots \text{(P-IV)} \\
 I &= \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x} \, dx \quad \dots \dots (2) \quad \begin{cases} \cos(\pi-x) = -\cos x \\ \sin(\pi-x) = \sin x \end{cases} \\
 (1) + (2) & \\
 2I &= \int_0^\pi \frac{x \sin x + \pi \sin x - x \sin x}{1+\cos^2 x} \, dx \\
 2I &= \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x} \, dx \\
 \text{Put } \cos x &= t \quad \text{when } x = 0 ; t = 1 \\
 \therefore \sin x \, dx &= -dt \quad \text{when } x = \pi ; t = -1 \\
 \therefore 2I &= -\pi \int_1^{-1} \frac{dt}{1+t^2} \\
 2I &= -\pi [\tan^{-1} t]_1^{-1} \\
 2I &= -\pi [\tan^{-1}(-1) - \tan^{-1}(1)] \\
 2I &= -\pi \left[-\frac{\pi}{4} - \frac{\pi}{4} \right] \\
 2I &= -\pi \left(-\frac{\pi}{2} \right) \\
 \therefore I &= \frac{\pi^2}{4} \quad \text{ans.}
 \end{aligned}$$

Q.4) $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} \, dx$

$$\begin{aligned}
 \text{Sol.4)} \quad I &= \int_0^\pi \frac{x \tan x}{\sec x + \tan x} \, dx \quad \dots \dots \text{[change in } \sin x \text{ & } \cos x\text{]} \\
 I &= \int_0^\pi \frac{x \sin x}{1 + \sin x} \, dx \quad \dots \dots (1)
 \end{aligned}$$



	$I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)} dx \quad \dots\dots(P-IV)$ $I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x} dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^{\pi} \frac{x\sin x + \pi\sin x - x\sin x}{1+\sin x} dx$ $2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$ <i>Type: rationalize</i> $2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} dx$ $2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx$ $2I = \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} dx$ $2I = \pi \int_0^{\pi} \tan x \sec x - \tan^2 x dx$ $2I = \pi \int_0^{\pi} \tan x \sec x - (\sec^2 x - 1) dx$ $2I = \pi [\sec x - \tan x + x]_0^{\pi}$ $2I = \pi [(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)]$ $2I = \pi [(-1 - 0 + \pi) - (1 - 0)]$ $2I = \pi[\pi - 2]$ $\therefore I = \frac{\pi}{2}(\pi - 2) \quad \text{ans.}$
Q.5)	$\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$
Sol.5)	$I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx \quad \dots\dots(1)$ $I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin\left(\frac{\pi}{2}-x\right) \cdot \cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right) + \cos^4\left(\frac{\pi}{2}-x\right)} dx \quad \dots\dots(P-IV)$ $I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \cos x \cdot \sin x}{\cos^4 x + \sin^4 x} dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x + \left(\frac{\pi}{2}-x\right) \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ $2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx$ Divide N & D by $\cos^4 x$ $2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$ Put $\tan^2 x = t$ when $x = 0 ; t = 0$ $2 \tan x \sec^2 x dx = dt$ when $x = \frac{\pi}{2} ; t = \infty$



	$\tan x \sec^2 x \ dx = \frac{dt}{2}$ $\therefore 2I = \frac{\pi}{4} \int_0^\infty \frac{dt}{t^2 + 1}$ $2I = \frac{\pi}{4} (\tan^{-1} t)_0^\infty$ $2I = \frac{\pi}{4} [\tan^{-1}(\infty) - \tan^{-1}(0)]$ $2I = \frac{\pi}{4} \left[\frac{\pi}{2} - 0 \right]$ $\Rightarrow 2I = \frac{\pi^2}{8}$ $\Rightarrow I = \frac{\pi^2}{16} \text{ ans.}$
Q.6)	$I = \int_0^\pi \frac{x \ dx}{a^2 \cos^2 x + b^2 \sin^2 x}$
Sol.6)	$I = \int_0^\pi \frac{x \ dx}{a^2 \cos^2 x + b^2 \sin^2 x} \quad \dots\dots\dots(1)$ $I = \int_0^\pi \frac{(\pi - x) \ dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} \quad \dots\dots\dots(\text{P-IV})$ $I = \int_0^\pi \frac{x \ dx}{a^2 \cos^2 x + b^2 \sin^2 x} \quad \dots\dots\dots(2)$ $(1) + (2)$ $2I = \int_0^\pi \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ $2I = \pi \int_0^\pi \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ <p>Type: Divide by $\cos^2 x$</p> <p>Divide N & D by $\cos^2 x$</p> $\therefore 2I = \pi \int_0^\pi \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$ $2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} \quad \dots\dots \left[\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \right] \quad \dots\dots(\text{P-VI})$ <p>Put $\tan x = t \quad x = 0 ; t = 0$</p> $\sec^2 x dx = dT \quad x = \frac{\pi}{2} ; t = \infty$ $\therefore 2I = 2\pi \int_0^\infty \frac{dt}{a^2 + b^2 t^2}$ $I = \frac{\pi}{b^2} \int_0^\infty \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$ $I = \frac{\pi}{b^2} \times \frac{b}{a} \left[\tan^{-1} \left(\frac{bt}{a} \right) \right]_0^\infty$ $I = \frac{\pi}{ab} [\tan^{-1}(\infty) - \tan^{-1}(0)]$ $I = \frac{\pi}{ab} \left[\frac{\pi}{2} - 0 \right]$ $I = \frac{\pi^2}{2ab} \text{ ans.}$
Q.7)	$I = \int_0^\pi \frac{x}{1 - \cos \alpha \sin x} dx$



Sol.7)	$I = \int_0^{\pi} \frac{x}{1-\cos \alpha \sin x} dx \quad \dots\dots(1)$ $I = \int_0^{\pi} \frac{\pi-x}{1-\cos \alpha \sin(\pi-x)} dx \quad \dots\dots(P-IV)$ $I = \int_0^{\pi} \frac{\pi-x}{1-\cos \alpha \sin x} dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^{\pi} \frac{x+\pi-x}{1-\cos \alpha \sin x} dx$ $2I = \pi \int_0^{\pi} \frac{1}{1-\cos \alpha \sin x} dx$ <p>(Type: single $\sin x, \cos x$)</p> $2I = \pi \int_0^{\pi} \frac{1}{1-\cos \alpha \cdot \frac{2 \tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}} dx$ $2I = \pi \int_0^{\pi} \frac{1+\tan^2 x}{1+\tan^2 \frac{x}{2}-2 \cos \alpha \cdot \tan \frac{x}{2}} dx$ $2I = \pi \int_0^{\pi} \frac{1+\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}-2 \cos \alpha \cdot \tan \frac{x}{2}+1} dx$ <p>Put $\tan \frac{x}{2} = t$ when $x = 0 ; t = 0$</p> $\sec^2 \frac{x}{2} \cdot dx = 2 dt \quad x = \pi ; t = \infty$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 - \frac{2t}{\cos \alpha} + 1}$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 - 2 \cos \alpha t + 1} \quad (\text{perfect square})$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{1}{(t-\cos \alpha)^2 - \cos^2 \alpha + 1} dt$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{(t-\cos \alpha)^2 - \sin^2 \alpha} dt \quad \dots\dots [1 - \cos^2 \alpha = \sin^2 \alpha]$ $2I = \frac{\pi}{2} \times \frac{1}{\sin \alpha} \left[\tan^{-1} \left(\frac{t-\cos \alpha}{\sin \alpha} \right) \right]_0^{\infty}$ $2I = \frac{\pi}{2 \sin \alpha} \left[\tan^{-1}(\infty) - \tan^{-1} \left(\frac{-\cos \alpha}{\sin \alpha} \right) \right]$ $I = \frac{\pi}{4} \sin \alpha \left[\frac{\pi}{2} - \tan^{-1}(-\cot \alpha) \right]$ $I = \frac{\pi}{4} \sin \alpha \left[\frac{\pi}{2} + \tan^{-1}(\cot \alpha) \right] \quad \dots\dots [\tan^{-1}(-x) = -\tan^{-1}x]$ $I = \frac{\pi}{4} \sin \alpha \left[\frac{\pi}{2} + \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \alpha \right) \right) \right]$ $I = \frac{\pi}{4 \sin \alpha} \left[\frac{\pi}{2} + \frac{\pi}{2} - \alpha \right]$ $I = \frac{\pi}{4 \sin \alpha} [\pi - \alpha] \quad \text{ans.}$
Q.8)	$I = \int_0^{\infty} \frac{\log x}{1+x^2} dx$
Sol.8)	$I = \int_0^{\infty} \frac{\log x}{1+x^2} dx$



	<p>Put $x = \tan \theta$ when $x = 0 ; \theta = 0$ $dx = \sec^2 \theta d\theta$ when $x = \infty ; \theta = \frac{\pi}{2}$</p> $\therefore I = \int_0^{\frac{\pi}{2}} \frac{\log(\tan \theta)}{1+\tan^2 \theta} \cdot \sec^2 \theta d\theta$ $I = \int_0^{\frac{\pi}{2}} \frac{\log(\tan \theta)}{\sec^2 \theta} \cdot \sec^2 \theta d\theta$ $I = \int_0^{\frac{\pi}{2}} \log(\tan \theta) d\theta \quad \dots\dots(1)$ <p>Proceed Yourself</p> <p>0 ans.</p>
Q.9)	$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$
Q.9)	$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$ <p>Put $x = \tan \theta$ when $x = 0 ; \theta = 0$ $dx = \sec^2 \theta d\theta$ when $x = 1 ; \theta = \frac{\pi}{4}$</p> $\therefore I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2 \theta} \cdot \sec^2 \theta d\theta$ $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) d\theta \quad \dots\dots(1)$ <p>Proceed yourself</p> <p>$I = \frac{\pi}{8} \log 2$ ans.</p>
Q.10)	$I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx$
Sol.10)	$I = \int_0^{\frac{\pi}{2}} \log(\sin x) dx \quad \dots\dots(1)$ $I = \int_0^{\frac{\pi}{2}} \log \left(\sin \left(\frac{\pi}{2} - x \right) \right) dx \quad \dots\dots(P-IV)$ $I = \int_0^{\frac{\pi}{2}} \log(\cos x) dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cdot \cos x) dx$ $2I = \int_0^{\frac{\pi}{2}} \log \left(\frac{\sin(2x)}{2} \right) dx$ $2I = \int_0^{\frac{\pi}{2}} \log(\sin(2x)) - \log 2 dx$ $2I = \int_0^{\frac{\pi}{2}} \log(\sin(2x)) dx - \int_0^{\frac{\pi}{2}} \log 2 dx$ $2I = \int_0^{\frac{\pi}{2}} \log(\sin(2x)) dx - \log 2 (x)_0$

$$2I = \int_0^{\frac{\pi}{2}} \log(\sin(2x)) dx - \frac{\pi}{2} \log 2$$

$$2I = I_1 - \frac{\pi}{2} \log 2 \quad \dots\dots\dots(3)$$

Where $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin(2x)) dx$

Put $2x = t$ when $x = 0 ; t = 0$

$$dx = \frac{dt}{2} \quad x = \frac{\pi}{2} ; t = \pi$$

$$\therefore I_1 = \frac{1}{2} \int_0^{\pi} \log(\sin t) dt$$

$$I_1 = \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log(\sin t) dt \quad \dots\dots\dots(P-VI)$$

$$I_1 = \int_0^{\frac{\pi}{2}} \log(\sin t) dt$$

$$I_1 = \int_0^{\frac{\pi}{2}} \log(\sin x) dx \quad \dots\dots\dots(P-I)$$

$$I_1 = I$$

\therefore eq. (3) becomes

$$2I = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2 \text{ ans.}$$