DEFINITE INTEGRALS

	Properties of Definite Integrals :-
P-I	$\int_a^b f(x)dx = \int_a^b f(t)dt$ e.g $\int_0^{\frac{\pi}{2}} \sin t\, dt = \int_a^{\frac{\pi}{2}} \sin x\, dx$
P-II	$\int_a^b f(x)dx = \int_{-b}^{a} f(x)dx$
P-III	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ Where $a < c < b$ e.g $\int_0^a f(x)dx = \int_a^{2a} f(x)dx + \int_0^{2a} f(x)dx$
P-IV	$\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Proof: Taking RHS $\int_0^a f(a-x)dx$ Put $a - x = t$ when $x = 0 \Rightarrow t = a$ $-dx = dt \Rightarrow dx = -dt$ when $x = a \Rightarrow t = a$ ∴ RHS = $-\int_a^0 f(t)dt$ = $\int_0^a f(t)dt$ ......(by P–II) = $\int_0^a f(x)dx$ ......(by P–I) = LHS ∴ $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ proved
P-V	$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ Proof: Do yourself by put $a + b - x = t$
P-VI	$\int_b^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \quad f(2a-x) = f(x) \\ 0 & ; \quad if \ f(2a-x) = -f(x) \end{cases}$ Mainly $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$
P-VII	**Even - function property** $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & ; \quad if \ f(x) \rightarrow \text{even} \\ 0 & ; \quad if \ f(x) \rightarrow \text{odd} \end{cases}$ If $f(-x) = f(x)$ then $f(x)$ is an even function If $f(-x) = -f(x)$ then $f(x)$ is an odd function
Q.1)	Evaluate $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx.$
Sol.1)	$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ....(1)

	$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$ .... $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$ ....(2) Adding (1) & (2) $2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ $= \int_0^{\frac{\pi}{2}} 1\,.dx$ $= (x)_0^{\frac{\pi}{2}}$ $2I = \frac{\pi}{2}$ $I = \frac{\pi}{4}$ ans.
Q.2)	Evaluate $I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x)\ dx$
Sol.2)	$I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x)\ dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \sin\left[2\left(\frac{\pi}{2}-x\right)\right].\log\left[\tan\left(\frac{\pi}{2}-x\right)\right]\ dx$ ....$\left[\int_0^a f(x)dx = \int_0^a f(a-x)\right]$ $I = \int_0^{\frac{\pi}{2}} \sin(\pi-2x).\log(\cot x)dx$ $I = \int_0^{\frac{\pi}{2}} \sin(2x).\log(\cot x)dx$ ....(2) $[\because \sin(\pi-2x) = \sin(2x)]$ Adding $eq.$ (1) & (2) $2I = \int_0^{\frac{\pi}{2}} \sin(2x)[\log(\tan x)+\log(\cot x)]\,dx$ $2I = \int_0^{\frac{\pi}{2}} \sin(2x)\log(\tan x\ .\cot x)\ dx$ $2I = \int_0^{\frac{\pi}{2}} \sin(2x)\,.\log(1)\ dx$ $2I = \int_0^{\frac{\pi}{2}} 0\,dx$ ....$\{\because \log 1 = 0\}$ $\therefore\ I = 0$ ans......
Q.3)	$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$
Sol.3)	$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\ dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$ .....$\left[\int_0^a f(x)dx = \int_0^a f(a-x)\ dx\right]$ $I = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]\ dx$ .....$\{\tan(A-B)\ formula\}$ $I = \int_0^{\frac{\pi}{4}} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]\ dx$

	$I = \int_0^{\frac{\pi}{4}} log\left(\frac{2}{1+\ tanx}\right)\ dx$ .....(2) $Eq.(1) + (2)$ $2\,I = \int_0^{\frac{\pi}{4}} log\left(1 + tan\,x\ .\frac{2}{1+tanx}\right)\ dx$ $2\,I = \int_0^{\frac{\pi}{4}} log\ (2)\ dx$ $2\,I = log\,2\,(x)_0^{\frac{\pi}{4}}$ $2\,I = log\,2\left[\frac{\pi}{4} - 0\right]$ $\therefore\ I = \frac{\pi}{8}log\,2\quad ans.$
Q.4)	$I = \int_0^{\frac{\pi}{2}} 2\log(\cos x) - \log(\sin(2x))dx$
Sol.4)	$I = \int_0^{\frac{\pi}{2}} 2\ log(cos\ x) - log(sin(2x))dx$ $I = \int_0^{\frac{\pi}{2}} log(cos^2x) - log\big(sin(2x)\big)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left(\frac{cos^2x}{sin(2x)}\right)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left(\frac{cos^2x}{sin(2x)}\right)\ dx$ $I = \int_0^{\frac{\pi}{2}} log\left[\frac{cot\ x}{2}\right] dx$ ... ... (1) $I = \int_0^{\frac{\pi}{2}} \log\left[\frac{\cot(\frac{\pi}{2}-x)}{2}\right] dx$ ......$\left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$ $I = \int_0^{\frac{\pi}{2}} log\ \left(\frac{tan\,x}{2}\right)$ ......(2) $Eq.(1) + (2)$ $2\,I = \int_0^{\frac{\pi}{2}} log\left(\frac{cot\ x}{2}\ .\frac{tan\ x}{2}\right)\ dx$ $2\,I = \int_0^{\frac{\pi}{2}} log\left(\frac{1}{4}\right) dx$ ... ... $\{\tan x\,.\cot x\ =\ 1\}$ $2\,I = \int_0^{\frac{\pi}{2}} log(1) - log(4)dx$ $2\,I = \int_0^{\frac{\pi}{2}} - log\ 4\ dx$ ... ... $\{\log 1\ =\ 0\}$ $2\,I = -log\ 4\ [x]_0^{\frac{\pi}{2}}$ $2\,I = -log\ 4\left[\frac{\pi}{2}\right]$ $I = -\frac{\pi}{4}\ log\,4\quad ans.$ (or) $I = \frac{-\pi}{4}\log(2)^2$ $I = \frac{-\pi}{2}\log 2\quad ans.$

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Q.5)	$I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx$
Sol.5)	$I = \int_0^1 log\left(\frac{1-x}{x}\right) dx$ ....(1) $= \int_0^1 log\left[\frac{1-(1-x)}{1-x}\right] dx$ ....$[\int_0^a f(x)dx = \int_0^a f(a-x)dx]$ $= \int_0^1 log\left[\frac{x}{1-x}\right] dx$ ....(2) (1) + (2) $2I = \int_0^1 \log\left(\frac{1-x}{x} \cdot \frac{x}{1-x}\right) dx$ $= \int_0^1 \log(1)\, dx$ $2I = 0$ ... . $\{\because \log 1 = 0\}$ $I = 0$ ans.
Q.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{5-x}} dx$
Sol.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{5-x}} dx$ ... ... (1) $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x}+\sqrt[3]{5-(5-x)}} dx$ ... ... $[\int_0^a f(x)dx = \int_0^a f(a-x)]$ $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x}+\sqrt[3]{x}} dx$ ....(2) (1) + (2) $2I = \int_0^5 \frac{\sqrt[3]{x}+\sqrt[3]{5-x}}{\sqrt[3]{x}+\sqrt[3]{5-x}} dx$ $= \int_0^5 1.dx$ $= (x)_0^5$ $2I = 5$ $I = \frac{5}{2}$ ans.
Q.7)	Show that $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
Sol.7)	R.H.S $\int_0^a f(x)dx + \int_0^a f(2a-x)dx$ Put $2a - x = t$ when $x = 0$; $t = 2a$ $-dx = dt$ $dx = -dt$ when $x = a$; $t = a$ $\therefore R.H.S = \int_0^a f(x)dx - \int_{2a}^a f(t)dt$ $= \int_0^a f(x)dx + \int_a^{2a} f(t)\, dt$ ......$\left[\int_0^b f(x)dx = -\int_b^a f(x)\right]$ $= \int_0^a f(x)dx + \int_a^{2a} f(x)dx$ ....$\left[\int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$= \int_0^{2a} f(x)dx$ ……$\left[\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx\right]$

$= LHS$ Proved

Q.8) Show that $I = \int_0^1 x(1-x)^n\, dx$

Sol.8)

$I = \int_0^1 (1-x)[1-(1-x)]^n dx$ ……$[\int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$I = \int_0^1 (1-x)(x)^n dx$

$I = \int_0^1 x^n - x^{n+1} dx$

$I = \left[\frac{x^n}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1$

$I = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] - [0-0]$

$I = \frac{n+2-n-1}{(n+1)\,(n+2)}$

$I = \frac{1}{(n+1)\,(n+2)}$ ans.

Q.9) $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$

Sol.9)

$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$ ………(1)

$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)} dx$ …….$(P-IV)$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx$ ……..(2)

(1) + (2)

$2\,I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$

$(Type{:}-single\ sin\ x\ \&\ cos\ x)$

$2\,I = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{2\tan x}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$

$2\,I = \int_0^{\frac{\pi}{2}} \frac{1+tan^2\left(\frac{x}{2}\right)}{2\,tan\frac{x}{2} + 1-tan^2\frac{x}{2}} dx$

$2\,I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{2\tan\frac{x}{2} + 1-\tan^2\left(\frac{x}{2}\right)} dx$

Put $\tan\left(\frac{x}{2}\right) = t$ when $x = 0$; $\tan(0) = t$

$\sec^2\left(\frac{x}{2}\right) . \frac{1}{2}\, dx = dt$ $t = 0$

$\text{Sec}^2\left(\frac{x}{2}\right) dx = 2dt$ when $x = \frac{\pi}{2}$; $\tan\left(\frac{\pi}{4}\right) = t$,

$t = 1$

	$\therefore\ 2I = 2\int_0^1 \frac{dt}{-t^2+2t+1}$ $I = -\int_0^1 \frac{1}{t^2-2t-1}\,dt$ $= -\int_0^1 \frac{1}{(t-1)^2-1-1}\,dt$ $= -\int_0^1 \frac{1}{(t-1)^2-(\sqrt{2})^2}\,dt$ $= \int_0^1 \frac{1}{(\sqrt{2})^2-(t-1)^2}\,dt$ $= \frac{1}{2\sqrt{2}}\left[log\left\|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right\|\right]_0^1$ $= \frac{1}{2\sqrt{2}}\left[log\left\|\frac{\sqrt{2}-0}{\sqrt{2}+0}\right\| - log\left\|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right\|\right]$ $= \frac{1}{2\sqrt{2}}\left[log(1) - log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$ $I = \frac{-1}{2\sqrt{2}} log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$ $ans.$ (Or) $I = \frac{-1}{2\sqrt{2}} log\left[\frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right]$ $\ldots\ldots\{Rationalize\}$ $= \frac{-1}{2\sqrt{2}} log\left[\frac{(\sqrt{2}-1)^2}{2-1}\right]$ $= -\frac{2}{2\sqrt{2}} log(\sqrt{2}-1)$ $I = \frac{-1}{\sqrt{2}}\log(\sqrt{2}-1)$ ans.
Q.10)	$I = \int_0^{\frac{\pi}{2}} \frac{cos^2 x}{1+sin\,x\,.cos\,x}\,dx$
Sol.10)	$I = \int_0^{\frac{\pi}{2}} \frac{cos^2 x}{1+sin\,x\,.cos\,x}\,dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \frac{cos^2\left(\frac{\pi}{2}-x\right)}{1+sin\left(\frac{\pi}{2}-x\right).cos\left(\frac{\pi}{2}-x\right)}\,dx$ $I = \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{1+cos\,x\,.\,sin\,x}\,dx$ ....(2) $(1)\ +\ (2)$ $2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+sin\,x\,.cos\,x}\,dx$ ....$\{sin^2 x + \cos^2 x = 1\}$ $Type: Divide\ by\ cos^2 x$ $Divide\ N\ \&\ D\ by\ cos^2 x$ $2I = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{sec^2 x + tan\,x}\,dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{sec^2 x}{1 + tan^2 + tan\,x}\,dx$

Put $tan\,x = t$ when $x = 0$; $t = 0$

$\therefore\ \sec^2 x.dx = dt$ when $x = \frac{\pi}{2}$; $t = \infty$

$$\therefore\ 2\,I = \int_0^\infty \frac{dt}{t^2+t+1}$$

Perfect square

$$2\,I = \int_0^\infty \frac{1}{\left(t+\frac{1}{2}\right)^2-\frac{1}{4}+1}dt$$

$$2\,I = \int_0^\infty \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2\,I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^\infty$$

$$2\,I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^\infty$$

$$2\,I = \frac{2}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$2\,I = \frac{2}{\sqrt{2}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$2\,I = \frac{2}{\sqrt{2}}\left[\frac{\pi}{3}\right]$$

$I = \frac{\pi}{3\sqrt{3}}$ ans.