Determinants
Class 12th

Q.1)	Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab$
Sol.1)	We have $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ taking a , b , c common from R_1 , R_2 & R_3 respectively $= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$ $= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$ $c_2 \to c_2 - c_1$ and $c_3 \to c_3 - c_1$ $= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$ expanding along R_1 $= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \times 1$ $= abc \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ = RHS $= abc + bc + ca + ab$ = RHS ans.
Q.2)	Show $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$
Sol.2)	We have $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ taking a common from C_1 $= \begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$ $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$= a \begin{vmatrix} 1 & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

$R_3 \to R_3 - 3R_2$

$$= a \begin{vmatrix} 1 & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

expanding along R_1

$= a[a^2] = a^3$ ans.

Q.3)

If x , y , z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ then show $xyz = -1$.

Sol.3)

We have $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

applying sum property in C_3

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$

taking x , y , z common R_1 , R_2 , R_3 resp.

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0$

$c_2 \leftrightarrow c_3$

$\Rightarrow - \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$

$c_1 \leftrightarrow c_2$

$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} (1 + xyz) = 0$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} (1 + xyz) = 0$

$\Rightarrow (y-x)(z-x) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} (1 + xyz) = 0$

expanding along R_1

$\Rightarrow (y-z)(z-x)[z+x-y-x](1+xyz) = 0$

$\Rightarrow (y-x)(z-x)(z-y)(1+xyz) = 0$

but $\Rightarrow y - x \neq 0$

$z - x \neq 0$ since $x \neq y \neq z$ given

$z - y \neq 0$

	$\therefore$ only $1 + xyz = 0$ $\Rightarrow xyz = -1$ Proved
Q.4)	Show $\begin{vmatrix} -bc & b^2 + bc & c^2 + bc \\ a^2 + ac & -ac & c^2 + ac \\ a^2 + ab & b^2 + a & b - ab \end{vmatrix} = (ab + bc + ca)^3$
Sol.4)	$R_1 \to aR_1; R_2 \to bR_2 \text{ and } R_3 \to cR_3$ $= \dfrac{1}{abc} \begin{vmatrix} -abc & ab^2 + abc & ac^2 + abc \\ a^2b + abc & -abc & c^2b + abc \\ a^2c + abc & b^2c + abc & -abc \end{vmatrix}$ taking a , b , c common from $c_1, c_2 \text{ and } c_3$ $= \dfrac{abc}{abc} \begin{vmatrix} -bc & ab + ac & ac + ab \\ ab + bc & -ac & bc + ab \\ ac + bc & bc + ac & -ab \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$ $= \begin{vmatrix} ab + bc + ac & ab + bc + ac & ab + bc + ac \\ ab + bc & -ac & bc + ab \\ ac + bc & bc + ac & -ab \end{vmatrix}$ taking $(ab + bc + ca)$ common from R$_1$ $= (ab + bc + ca) \begin{vmatrix} 1 & 1 & 1 \\ ab + bc & -ac & bc + ab \\ ac + bc & bc + ac & -ab \end{vmatrix}$ $c_2 \to c_2 - c_1 \text{ and } c_3 \to c_3 - c_1$ $= (ab + bc + ca) \begin{vmatrix} 1 & 0 & 0 \\ ab + bc & -ab - bc - ac & 0 \\ ac + bc & 0 & -ab - bc - ca \end{vmatrix}$ taking (ab + bc + ca) common from $c_2 \text{ and } c_3$ both $= (ab + bc + ca)(ab + bc + ca)^2 \begin{vmatrix} 1 & 0 & 0 \\ ab + bc & -1 & 0 \\ ac + bc & 0 & -1 \end{vmatrix}$ expanding along R$_1$ $= (ab + bc + ca)^3 (1) = (ab + bc + ca)^3 = $ RHS
Q.5)	Show $\begin{vmatrix} b^2 + c^2 & ab & ac \\ ba & c^2 + b^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix} = 4a^2 b^2 c^2$
Sol.5)	We have $\begin{vmatrix} b^2 + c^2 & ab & ac \\ ba & c^2 + b^2 & bc \\ ca & cb & a^2 + b^2 \end{vmatrix}$ $R_1 \to aR_1; R_2 \to bR_2 \text{ and } R_3 \to cR_3$ $= \dfrac{1}{abc} \begin{vmatrix} a(b^2 + c^2) & a^2b & a^2c \\ ab^2 & b(c^2 + b^2) & b^2c \\ c^2a & c^2b & c(a^2 + b^2) \end{vmatrix}$ taking a , b , c common from c_1, c_2, c_3 resp. $= \dfrac{abc}{abc} \begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$ $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(b^2 + c^2) & 2(c^2 + a^2) & 2(a^2 + b^2) \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

2 common from R_1

$$= \begin{vmatrix} b^2 + c^2 & c^2 + a^2 & a^2 + b^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2 \begin{vmatrix} b^2 + c^2 & c^2 + a^2 & a^2 + b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= 2 \begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -b^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

expanding

$$= 2[-c^2(-a^2 b^2) + b^2(a^2 c^2)]$$
$$= 2(a^2 b^2 c^2 + a^2 b^2 c^2) = 4a^2 b^2 c^2 \quad \text{ans.}$$

Q.6)

Show that $\begin{vmatrix} b + c & c + a & a + b \\ q + r & r + p & p + q \\ y + z & z + x & x + y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

Sol.6)

We have $\begin{vmatrix} b + c & c + a & a + b \\ q + r & r + p & p + q \\ y + z & z + x & x + y \end{vmatrix}$

$c_1 \to c_1 + c_2 + c_3$

$$= \begin{vmatrix} 2(a + b + c) & c + a & a + b \\ 2(a + b + c) & r + p & p + q \\ 2(x + y + z) & z + x & x + y \end{vmatrix}$$

$$= 2 \begin{vmatrix} a + b + c & c + a & a + b \\ p + q + r & r + p & p + q \\ x + y + z & z + x & x + y \end{vmatrix}$$

$c_2 \to c_2 - c_1$ and $c_3 \to c_3 - c_1$

$$= 2 \begin{vmatrix} a + b + c & -b & -c \\ p + q + r & -q & -r \\ x + y + z & -y & -z \end{vmatrix}$$

Now, $c_1 \to c_1 + c_2 + c_3$

$$= 2 \begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix}$$

taking (−) sign from c_1 & c_3 both

$$= 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{RHS}$$

Q.7)	Show $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$
Sol.7)	We have $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ $R_1 \to R-1-xR_2$ $= \begin{vmatrix} a-ax^2 & c-cx^2 & p-px^2 \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ $= \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ taking $(1-x^2)$ common from R_1 $= (1-x^2) \begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$ $= R_2 \to R_2 - xR-1$ $= (1-x^2) \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$ = RHS
Q.8)	Show that the value of the determinants $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.
Sol.8)	let $\Delta \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ $c_1 \to c_1 + c_2 + c_3$ $= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a^2+b+c & a & b \end{vmatrix}$ (a + b + c) common from C_1 $= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ $= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ expanding along R_1 $= (a+b+c)(-a^2-b^2-c^2+ab+bc+ca)$ $= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ multiply & divide by 2 $= -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$ $= -\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$ clearly the value of determinant is –ve ans.

Q.9)	If a , b , c are real number such that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ then show that either a + b + c = 0 (or) a = b = c.
Sol.9)	We have $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ $c_1 \rightarrow c_1 + c_2 + c_3$ $\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0$ $\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+b \end{vmatrix} = 0$ $R_2 \rightarrow$ and $R_3 \rightarrow R_3 - R_1$ $\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} = 0$ expanding along R₁ $\Rightarrow 2(a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) = 0$ $\Rightarrow -2(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$ multiply and divide by 2 $\Rightarrow -\frac{2}{2}(a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = 0$ $\Rightarrow -(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$ $\Rightarrow (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$ $\Rightarrow$ either $a + b + c = 0$ (or) $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$ this is possible only when $a - b = 0 \Rightarrow a = b$ $b - c = 0 \Rightarrow b = c$ $c - a = \Rightarrow c = a$ $\Rightarrow a = b = c$ $\therefore$ either $a + b + c = 0$ (or) $a = b = c$ ans.
Q.10)	Show that $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$
Sol.10)	let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$ $R_1 \rightarrow cR_1 ; R_2 \rightarrow bcR_2$ and $R_3 \rightarrow aR_3$ $= \frac{1}{abc} \begin{vmatrix} 0 & ac & -bc \\ -a & b0 & -bc \\ ab & ac & 0 \end{vmatrix}$ taking ab , ac , bc common from C₁ , C₂ & C₃ $= \frac{(ab)(ac)(bc)}{abc} \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}$

$$= abc \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

expanding

$= abc(0) = 0 =$ RHS   ans.