CBSE Class 12 Mathematics Differentiation Worksheet

Continuity & Differentiability

Q1. For what value of λ is the function defined by

$f(x) = \{\lambda(x^2 - 2x);$ if $x \leq 0\}$

$\{4x + 1 \;;\; x < 0\}$

continues at x = 0 ? what about continuity at x = 1 ?

Sol.1 Continuity at x = 0

LHL = $(\lambda(x^2 2x))$

put x = 0 – h = –h and h → 0

LHL = $(\lambda(h^2 + 2h)) = \lambda(0 + 0)$

LHL = 0

RHL (4x + 1)

put x = 0 + h = h and h ← 0

∴ RHL = (4h + 1) = 1

RHL = 1

since LHL ≠ RHL

∴ f(x) is not continuous at x = 0 for any value of λ

Continuity at x = 1

here for LHL and RHL ; f(x) = 4x + 1 (same)

LHL = (4x + 1)

put x = 1 – h and h → 0

LHL = (4(1 – h) + 1) = 4 + 1

LHL = 5

RHL = (4x + 1)

put x = 1 + h and h → 0

RHL = (4(1 + h) + 1)

RHL = 4 + 1 = 5

f(1) = 4 (1) + 1

f(1) = 5

f(x) is continuous at x = 1 ; irrespective of any value of λ Ans.

Q2. Discuss the continuity of f(x) given by

f (x) = $\left\{x^2 \sin\left(\frac{1}{x}\right); \text{if} x \neq 0\right\}$

{0 ; x = 0}

Sol.2 x^2 polynomial function which is everywhere continuous

and sin(1/x) is a sine function which is also everywhere continuous

and product of two continuous function is also continuous

when $x \neq 0$

f(x) = $x^2 \sin(1/x)$

$\therefore$ f(x) is continuous for all $x \neq 0$

continuity at x = 0

LHL = $\left(x^2 \sin\left(\frac{1}{x}\right)\right)$

put x = 0 – h = –h and h $\rightarrow$ 0

LHL = $\left((-h)^2 . \sin\left(\frac{-1}{h}\right)\right)$

= $\left(-h^2 . \sin\left(\frac{1}{h}\right)\right)$

= $\left(-h^2 . \sin\left(\frac{1}{h}\right)\right)$

= $0 \sin\left(\frac{1}{0}\right) = 0 \times$ (an oscillating number between -1 and 1)

LHL = 0

similarly RHL = 0 (Do Yourself)

and f(0) = 0

$\therefore$ LHL= RHL = f(0) = 0

$\therefore$ f(x) is also continuous at x = 0

$\therefore$ f(x) is continuous everywhere (or) there is no point of discontinuity. (Ans)

Q3. Prove that the function f(x) = | x -1 | $\in$ R is not differentiable at x = 1.

Sol.3. We have f(x) = |x – 1| = {(x – 1) ; x > 1}

{– (x – 1) ; x < 1}

LHD = $\left[\frac{f(x)-f(1)}{x-1}\right]$

= $\left[\frac{-(x-1)-(1-1)}{x-1}\right]$

= $\left[\frac{-(x-1)}{(x-1)}\right]$

LHD = (–1)

$\Rightarrow$ LHD = –1

RHD = $\left(\frac{f(x)-f(1)}{x-1}\right)$

= $\left(\frac{(x-1)-(1-1)}{x-1}\right)$

= $\left(\frac{(x-1)}{(x-1)}\right)$

RHD = (1)

$\therefore$ RHD = 1

$\because$ LHD $\neq$ RHD $\therefore$ f(x) is not Differentiable at x = 1 Ans.

Q4. Prove that the greatest integer function f(x) = [x] ; 0 < x < 3 is not differentiable at x = 1 and x = 2.

Sol.4. We have f(x) = [x] ; 0 < x < 3

f(x) = {0 ; 0 < x < 1}

{1 ; 1 $\leq$ x < 2}

{2 ; 2 $\leq$ x < 3

Differentiability at x = 1

LHD = $\left(\frac{f(x)-f(1)}{x-1}\right)$

= $\left[\frac{0-1}{x-1}\right]$

put x = 1 – h and h $\to$ 0

$\therefore$ LHD = $\left(\frac{-1}{1-h-1}\right)$

= $\left(\frac{1}{h}\right)$ = ∞ $\therefore$ LHD = ∞

RHD = $\left(\frac{f(x)-f(1)}{x-1}\right)$

= $\left(\frac{1-1}{x-1}\right)$

= $\left(\frac{0}{x-1}\right)$

RHD = 0

since LHD $\neq$ RHD

$\therefore$ f(x) is not Differentiability at x = 1

Similarly : check the Differentiability at x = 2

Q5. If f(x) = | x |^3. Show that f ^{11}(x) exists for all x $\in$ R and find it.

Sol.5 We have , $f(x) = |x|^3$

$\Rightarrow f(x) = |x| = \{x^3 \;;\; x \geq 0\}$

$\{-x^3 \;;\; x < 0\}$

LHD = $\left(\frac{f(x)-f(0)}{x-0}\right)$

$= \left(\frac{-x^3-0}{x}\right)$

$= (2x^2)$

put $x = 0 - h = -h$ and $h \to 0$

$\therefore$ LHD = $(-(-h)^2) = 0$

$\therefore$ LHD = 0

Now RHD = $\left(\frac{f(x)-f(0)}{x-0}\right)$

$= \left(\frac{x^3-0}{x-0}\right) = (x^2)$

put $x = 0 + h = h$ and $h \to 0$

$\therefore$ RHD = $(h^2) = 0$

RHD = 0

since LHD = RHD

$\therefore$ f(x) is differentiable at x = 0

$\therefore f^1(x)$ exists and given by

$f^1(x) = \{3x^2 \;;\; x \geq 0\}$

$\{-3x^2 \quad x < 0\}$

LHD = $\left(\frac{f^1(x)-f^1(0)}{x-0}\right)$

$= (-3x)$

put $x = 0 - h = -h$ and $h \to 0$

$\therefore$ LHD = (3h) = 0

$\therefore$ LHD = 0

Similarly RHD = 0

since LHD = RHD

$\therefore f^1(x)$ is differentiable at x = 0

$\therefore f^{11}(x)$ exists and given by

$f^{11}(x) = \{6x \;;\; x \neq 0\}$

$\{-6 \;;\; x < 0\}$

Q6. Find the values of 'a' and 'b' so that the function f(x) $\{x^2 + 3x + a \;;\; x \leq 1\}$ is differentiable

$\{bx + 2 \;;\; x > 1\}$

at x = 1.

Sol.6. Since f(x) is differentiable at x = 1

$\therefore$ f(x) is also continuous at x = 1

continuity at x = 1

RHL = (bx + 2)

put x = 1 + h and h $\rightarrow$ 0

LHL = (b(1 + h) + 2)

LHL = b + 2

LHL = $(x^2 + 3x + a)$

put x = 1 + h and h $\rightarrow$ 0

$\therefore$ RHL = $((1 + h)^2 + 3(1 + h) + a)$

RHL = 1 + 3 + a

LHL = 4 + a

f(1) = 1 + 3 + a = 4 + a

we have , RHL = LHL = f(1)

$\Rightarrow$ b + 2 = 4 + a = 4 + a

$\Rightarrow$ b + 2 = 4 + a

$\Rightarrow$ b = 2 + a ......(1)

Differentiability at x = 1

$$LHD = \left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \left(\frac{x^2 3x+a-(1+3+a)}{x-1}\right)$$

$$= \left(\frac{x^2+3x+a-4-a}{(x-1)}\right)$$

$$= \left(\frac{x^2+3x-4}{(x-1)}\right)$$

$$= \left(\frac{(x+4)(x-1)}{(x-1)}\right)$$

put x = 1- h and h $\rightarrow$ 0

LHD = (1 – h + 4)

LHD = 5

$$RHD = \left(\frac{bx+2-f(1)}{x-1}\right)$$

$= \left(\frac{bx+2-(4+a)}{x-1}\right)$

$= \left(\frac{bx+2-4-a}{x-1}\right)$

$= \left(\frac{bx-(2+a)}{x-1}\right)$

$= \left(\frac{bx-b}{x-1}\right)$ ……{from (1) b = a + 2}

$= \left(\frac{b(x-1)}{(x-1)}\right)$

RHL = b

since f(x) is differentiable at x = 1

$\therefore$ LHD = RHD

$\Rightarrow 5 = b$

$\therefore$ b = 5 put in (1)

a = 3 (Ans) a = 3 & b = 5

Q7. Show that f(x) is discontinues at x = 0

f(x) $\left\{\frac{e^{1/x}-1}{e^{1/x}+1}; x \neq 0\right\}$

{0 ; x = 0}

Sol.7 LHL = $\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$

put x = 0 – h = –h and h ⟶ 0

LHL = $\left(\frac{e^{-1/x}-1}{e^{-1/h}+1}\right)$

$= \left(\frac{e^{-\infty}+1}{e^{\infty}+1}\right)$

$= \frac{0-1}{0+1}$ ….$\{e^{\infty} = 0\}$

LHL = –1

RHL = $\left(\frac{e^{1/x}-1}{e^{1/x}+1}\right)$

put x = 0 + h and h ⟶ 0

RHL = $\left(\frac{e^{1/h}-1}{e^{1/h}+1}\right)$

$= \left(\frac{1-e^{-1/h}}{1+e^{-1/h}}\right)$ …..{Divide by $e^{1/h}$}

$= \frac{1-e^{-\infty}}{1+e^{-\infty}}$

$= \frac{1-0}{1+0}$

RHL = 1

since LHL ≠ RHL $\therefore$ f(x) is not continuous at x = 0. (Ans)

Rolle's And Mean Value Theorem

Q8. Verify mean value theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval a = 1 & b = 3 i.e. [1, 3]. Find all c (1 , 3) for which $f^1(c) = 0$.

Sol.8 We have , $f(x) = x^3 - 5x^2 - 3x$; x [1 , 3]

since f(x) is a polynomial function which is everywhere continuous.

$\therefore$ f(x) is continuous in [1 , 3]

Diff f(x) w.r.t. x

$f^1(x) = 3x^2 - 10x - 3$

clearly $f^1(x)$exists for all x ← (1 , 3)

$\therefore$ f(x) is differentiable in (1 , 3)

The two conditions all statistical then there exists a value c ← (1 , 3) such that

$f^1(c) = \frac{f(3)-f(1)}{3-1}$

$\Rightarrow 3c^2 - 10c - 3 = \frac{-20}{2}$

$\Rightarrow 3c^2 - 10c - 3 = -10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c-1) - 7(c-1) = 0$

c = 1 ; c = 7/3

clearly c = 7/3 ← (1 , 3)

$\therefore$ mean value theorem is verified

(ii) given $f^1(c) = 0$

$\Rightarrow 3c^2 - 10c - 3 = 0$

a = 3 ; b = –10 ; c = –3

$c = \frac{10 \pm \sqrt{100+36}}{6}$ {quadratic formula}

$c = \frac{10 \pm \sqrt{136}}{6}$

$\Rightarrow c = \frac{10 \pm 11.7}{6}$

c = 3.6 and c = –0.28

clearly both value does not belong to (1 , 3)

$\therefore$ there is no value of c for which $f^1(c) = 0$.

Q9. Verify Rolle's theorem for the function $f(x) = \sqrt{4 - x^2}$; on [-2 , 2].

Sol.9 We have , $f(x) = \sqrt{4 - x^2}$

$\rightarrow$ for all $x \leftarrow [-2, 2]$, the limit of the function is equal to the value of the function

i.e. limit $f(x) = f(a)$ where $a \leftarrow [-2, 2]$

$\therefore$ f(x) is continuous for all $x \leftarrow [-2, 2]$

Differentiable f(x) w.r.t. x

$f(x) = \frac{1}{2\sqrt{4-x^2}}(-2x) = -\frac{x}{\sqrt{4-x^2}}$

clearly $f^1(x)$ exists for all x (–2 , 2)

$\therefore$ f(x) is differentiable for all $x \leftarrow (-2, 2)$

$f(-2) = \sqrt{4-(-2)^2} = \sqrt{4-4} = 0$

$f(2) = \sqrt{4-2^2} = \sqrt{4-4} = 0$

$\therefore f(-2) = f(2)$

the there conditions of rolle's theorem are satisfied , then there exists a value

c (–2 , 2) such that $f^1(c) = 0$

Now $f(c) = \frac{-c}{\sqrt{4-c^2}}$

$\Rightarrow \frac{-c}{\sqrt{4-c^2}} = 0$

$\Rightarrow -c = 0$

$\Rightarrow c = 0$

clearly $c = 0 \leftarrow (-2, 2)$

hence Rolle's theorem is verified. (Ans.)

Q10. Discuss the “applicability” of Rolle's theorem on indicated intervals.

Sol.10 (i) $f(x) = 3 + (x-2)^{2/3}$ on [1 , 3]

$f^1(x) = 0 + \frac{2}{3}(x-2)^{-1/3}$

$f^1(x) = \frac{2}{3(x-2)^{1/3}}$

clearly $f^1(x)$ does not exists when x = 2

$\therefore$ f(x) is not differentiable in interval (1 , 3)

hence Rolle's theorem not applicable

(ii) $f(x) = \tan x$ on $[0, \pi]$

clearly $\tan\left(\frac{\pi}{2}\right) = \infty$

$\therefore$ tan x is not continuous at $x = \frac{\pi}{2}$

$\therefore$ f(x) is not continuous on $[0, \pi]$

Hence Rolle's theorem not applicable

(ii) $f(x) = [x]$; $x \leftarrow [-1, 1]$

$f(-1) = [-1] = -1$

and $f(1) = [1] = 1$

clearly $f(-1) \neq f(1)$

hence Rolle's theorem not applicable

(iv) $f(x) = |x|$; $x \leftarrow [-1, 1]$

we have that modulus function is not Diff at x = 0

$\therefore$ f(x) is not Diff in $(-1, 1)$

Hence Rolle's theorem not applicable

(v) $f(x) = \{-4x + 5 \ ; \ 0 \leq x \leq 1\}$

$\{2x - 3 \ ; \ 1 < x \leq 2\}$

here LHL = 1 & RHL = –1 (Do yourself)

clearly LHL $\neq$ RHL

$\therefore$ f(x) is not continuous at x = 1

$\therefore$ Rolle's theorem not applicable (Ans.)