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CBSE Class 12 Mathematics Differentiation Worksheet

Higher Order Derivative

Q1. If $y = \cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Sol.1 We have, $y = \cos^{-1}x$ -------(1)

Diff w.r.t x

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = -1 \quad \text{(cross thus SH e)}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Diff again w.r.t x (using Quotient rule)

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}(0)-(-1)\frac{1}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{(1-x^2)\sqrt{1-1x^2}}$$

from of (1) x = cos y, put in $\frac{d^2y}{dx^2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{(1-\cos^2 y)\sqrt{1-\cos^2 y}}$$

$$\Rightarrow \frac{-\cos y}{\sin^2 y.\sin y}$$

$$\frac{d^2y}{dx^2} = -\cot y.\,cosec^2 y \text{ Ans.}$$

Q2. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Sol.2 We have, $y = Ae^{mx} + Be^{nx}$ ------(1)

Diff w.r.t x,

$$\frac{dy}{dx} = mAe^{mx} + nBe^{nx} \text{------(2)}$$

Diff again w.r.t x

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{(1-\cos^2 y)\sqrt{1-\cos^2 y}} \quad \text{----(3)}$$

Taking LHS

$$\frac{d^2y}{dx^2} = (m+n)\frac{dy}{dx} + mny$$

Substituting the value of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ & y ...{from (1), (2) & (3)}

$$m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$$\Rightarrow m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx} - mnBe^{nx} - mnAe^{mx} - n^2Ae^{nx} + mnAe^{mx} + mnBe^{nx} = 0 \text{RHS.} \quad \text{(Proved)}$$

Q3. If y = tan x + sec x, show that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Sol.3 We have y = tan x + sec x

Diff w.r.t x

$$\frac{dy}{dx} = sec^2x + secx\tan x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-\cos y}{(1-\cos^2 y)\sqrt{1-\cos^2 y}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\sin x}{\cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\sin x}{(1-\sin x)(1+\sin x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1-\sin x}$$

Diff again w.r.t. x (using Quotient rule)

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(1-\sin x)(0)-1(-\cos x)}{(1-\sin x)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2} \qquad \text{(Proved)}$$

Q4.

If $(x-a)^2 + (y-b)^2 = c^2$-------(1) , Show that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant and independent of a and b.

Sol.4 We have , $(x-a)^2 + (y-b)^2 = c^2$ …..(1)

Diff w.r.t. x

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-a) + (y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-(x-a)}{(y-b)} \qquad \text{…..(2)}$$

Diff again w.r.t. x (Quotient rule on RHL)

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b)(1)-(x-a)dy/dx}{(y-b)^2}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b)+(x-a)\left(\frac{x-a}{y-b}\right)}{(y-b)^2}\right] \qquad \text{…..\{from eq. (2)\}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-(c^2)}{(y-b)^3} \qquad \text{……\{From eq. (1)\}}$$

Consider $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)}$

$$= \frac{\left(1+\frac{(x-a)^2}{(y-b)^2}\right)^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left(\frac{(x-a)^2+(y-b)^2}{(y-b)^2}\right)^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{3/2}}{\frac{-c^2}{(y-b)^3}}$$

$$= \frac{\frac{c^3}{(y-b)^3}}{\frac{-c^3}{(y-b)^3}} = \frac{c^3}{-c^3} = -c$$ which is a constant and independent of a and b. (Proved)

Q5. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$ find $\frac{d^2y}{dx^2}$.

Sol.5 We have ,

$$x = a(\theta - \sin\theta)$$

Diff w.r.t. θ

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad ....(1)$$

$$y = a(1 + \cos\theta)$$

Diff w.r.t θ

$$\frac{dy}{d\theta} = a(-\sin\theta)$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$\Rightarrow \frac{dy}{dx} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin(\theta/2).\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$\Rightarrow \frac{dy}{dx} = -\cot(\theta/2)$$

Now Diff w.r.t x

$$\frac{d^2y}{dx^2} = -\left[-cosec^2\frac{\theta}{2}\right].\frac{1}{2}.\frac{d\theta}{dx} \quad \{\text{main step}\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}cosec^2\left(\frac{\theta}{2}\right).\frac{1}{a(1-\cos\theta)} \quad .....\{\text{from eq. (1)}\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2a}.cosec^2\left(\frac{\theta}{2}\right).\frac{1}{2\sin^2(\theta/2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right) \quad \text{Ans.}$$

Q6. If $x = a(\cos\theta + \sin\theta)$ & $y = a(\sin\theta - \theta\cos\theta)$. Find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$.

Sol.6 We have, $x = a(\cos\theta + \theta\sin\theta)$

Diff w.r.t θ

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta.\cos\theta + \sin\theta]$$

$$\frac{dx}{d\theta} = a\theta\cos\theta \quad ......(1)$$

$$y = a(\sin\theta - \theta\cos\theta)$$

Diff w.r.t. θ

$$\frac{dy}{d\theta} = a(\cos\theta - (-\theta\sin\theta + \cos\theta))$$

$$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$\Rightarrow \frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Diff now w.r.t. 'x'

$$\frac{d^2y}{dx^2} = sec^2\theta.\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = sec^2\theta.\frac{1}{a\theta\cos\theta} = \frac{1}{a\theta}.sec^3\theta$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\pi/3} = \frac{1}{a.\frac{\pi}{3}}sec^3(\pi/3)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\pi/3} = \frac{3}{a\pi}(2)^3 \quad ....\left\{\cos\frac{\pi}{3} = 1/2 \therefore sec\,\pi/3 = 2\right\}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\pi/3} = \frac{24}{a\pi}$$

Ans.

Q7. Find A and B so that y = A sin(3x) + B cos(3x) satisfies the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10\cos(3x)$$

Sol.7 We have , y = A sin(3x) + B cos(3x)

Diff w.r.t. x

$$\frac{dy}{dx} = 3A.\cos(3x) - 3B\sin(3x)$$

Diff again w.r.t. x

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\pi/3} = \frac{1}{a.\frac{\pi}{3}}sec^3(\pi/3)$$

given equation :

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 10\cos(3x)$$

subst. The values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$

$$-9A\sin(3x) - 9B\cos(3x) + 4(3A\cos(3x) - 3B\sin(3x)) + 3(A\sin(3x) + B\cos(3x)) = 10\cos(3x)$$

$$\Rightarrow -9A\sin(3x) - 9B\cos(3x) + 12A\cos(3x) - 12B\sin(3x) + 3(A\sin(3x) + B\cos(3x)) = 10\cos(3x)$$

$$\Rightarrow \sin(3x)[-9A - 12B + 3A] + \cos(3x)[-9B + 12A + 3B] = 10\cos(3x)$$

$$\Rightarrow \sin(3x)(-6A - 12B) + \cos(3x)(-6B + 12A) = 10\cos(3x)$$

equating the coefficients of sin(3x) and cos(3x) on both sides.

We have , $-6A - 12B = 0$

and $-6B + 12A = 10$

solving these two equations we get

$A = \frac{2}{3}, B = \frac{-1}{3}$ Ans.

Q8. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Sol.8 We have,

$$x = a\cos\theta + b\sin\theta \qquad\qquad y = a\sin\theta - b\cos\theta$$

Diff w.r.t θ

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \qquad\qquad \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

$$\frac{dx}{d\theta} = -(a\sin\theta - b\cos\theta) \qquad\qquad \frac{dy}{d\theta} = x$$

$$\frac{dx}{d\theta} = -y$$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{x}{y}$

$\Rightarrow y\frac{dy}{dx} = -x$ ……(1)

Diff again w.r.t x (product rule on LHL.)

$$\Rightarrow y.\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = -1$$

$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-x}{y}\right) = -1$ ….{from eq. (1)}

$$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -y$$

$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = o$ (Proved)

Implicit Functions And General Differentiation

Q9. If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Sol.9 We have, $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Diff. w.r.t. x

$$\frac{1}{x^2+y^2}\cdot\left(2x+2y\frac{dy}{dx}\right) = 2.\frac{1}{1+\frac{y^2}{x^2}}\cdot\left[\frac{x.\frac{dy}{dx}-y(1)}{x^2}\right]$$

$$\Rightarrow \frac{2}{x^2+y^2}\left(x+y\frac{dy}{dx}\right) = 2\frac{x^2}{x^2+y^2}\left(\frac{xdy/dx-y}{x^2}\right)$$

$$\Rightarrow x+y\frac{dy}{dx} = x\frac{dy}{dx}-y$$

$$\Rightarrow x+y = x\frac{dy}{dx}-y\frac{dy}{dx}$$

$$\Rightarrow x+y = (x-y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \qquad \text{(Proved)}$$

Q10. If $x\sqrt{1+y}+y\sqrt{1+x} = 0$, show that $\frac{dy}{dx} = \frac{-1}{(x-1)^2}$

Sol.10 We have , $x\sqrt{1+y}+y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+x} = -y\sqrt{1+x}$$

squaring

$$\Rightarrow x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2+x^2y = y^2+y^2x$$

$$\Rightarrow x^2-y^2 = y^2x-x^2y$$

$$\Rightarrow (x+y)(x-y) = yx(y-x)$$

$$\Rightarrow (x+y)(x-y) = -yx(x-y)$$

$$\Rightarrow x+y = -yx$$

$$\Rightarrow y+yx = -x$$

$$\Rightarrow y(1+x) = -x$$

$$\Rightarrow y = -\frac{x}{1+x}$$

Diff w.r.t. x (Quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)(-1)-(-x)(1)}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1-x+x}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{(1+x)^2} \qquad \text{(Proved)}$$