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CBSE Class 12 Mathematics Differentiation Worksheet

Q1 If $y = a^x + e^x + x^x + x^a + a^a$. Find $\frac{dy}{dx}$ at x = a.

Sol.1 Let u = x^x

$\therefore y = a^x + e^x + u + x^a + a^a$

Diff w.r.t. X

$\frac{dy}{dx} = a^x.\log a + e^x + \frac{du}{dx} + ax^{a-1} + 0$ …..(i)

Now u = x^x

taking log both sides

log u = x log x

Diff wrt 'x' (product rule)

$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{x} + \log x$

$\frac{du}{dx} = x^x(1 + \log x)$

$\therefore$ equation (i) becomes

$\frac{dy}{dx} = a^x.\log a + e^x + x^x(1 + \log x) + ax^{a-1}$

put x = a

$\left(\frac{dy}{dx}\right)_{x=a} = +e^a + a^a(1 + \log a) + a.a^{a-1}$

$= a^a \log a + e^a + a^a + a^a.\log a + a^a$

$\frac{dy}{dx} e^a + 2a^a(1 + \log a)$ Ans.

Q2. If $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$. Find $\frac{dy}{dx}$.

Sol.2 We have

$y = \left(\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right)^{1/2}$

taking log both sides

$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-y) - \log(x-5)]$…..{using log proper}

Diff w.r.t. x

$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

$\frac{dy}{dx} - \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

$\therefore \frac{dy}{dx} - \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$ Ans.

Q3. (i) $y = (x^x)^x$ (ii) $y = x^{x^x}$. Find $\frac{dy}{dx}$.

Sol.3 (i) $y = (x)^x$

$\Rightarrow y = x^{x^2}$

taking log on both sides

$\log y = x^2 . \log x$

Diff wrt x

$\frac{1}{y} . \frac{dy}{dx} = x^2 . \frac{1}{x} + \log x . (2x)$

$\frac{dy}{dx} = y(x + 2x . \log x)$

$= x^x (x + 2x . \log x)$ Ans.

(ii) $y = x^{x^x}$

let $x^x = u$

$\therefore y = x^u$

taking log on both sides

log y = u log x

Diff w.r.t x

$\frac{1}{y} . \frac{dy}{dx} = u . \frac{1}{x} + \log x . \frac{du}{dx}$

$\Rightarrow \frac{dy}{dx} = y\left(\frac{u}{x} + \log x . \frac{du}{dx}\right)$

$\frac{dy}{dx} = x^{x^x}\left(\frac{x^x}{x} + \log x . \frac{du}{dx}\right)$ .....(1)

Now $u = x^x$

taking log on both sides

log u = x log x

Diff w.r.t x

$\frac{1}{u} . \frac{du}{dx} = x . \frac{1}{x} + \log x$

$\frac{du}{dx} = u(1 + \log x)$

$\frac{du}{dx} = x^x (1 + \log x)$

equation (i) becomes

$\frac{dy}{dx} x^{x^x}\left[\frac{x^x}{x} + \log x . x^x (1 + \log x)\right]$

$\therefore \frac{dy}{dx} = x^{x^X} . x^x \left(\frac{1}{x} + \log x . (1 + \log x)\right)$ Ans.

Q4. Given that $\cos\frac{x}{2} . \cos\frac{x}{4} . \cos\frac{x}{8} \ldots\ldots \infty = \frac{\sin x}{x}$

show that $\frac{1}{2^2} . sec^2\left(\frac{x}{2}\right) + \frac{1}{2^4} . sec^2\left(\frac{x}{4}\right) + \cdots \ldots = cosec^2 x - \frac{1}{x^2}$

Sol.4 We have $\cos\left(\frac{x}{2}\right) . \cos\left(\frac{x}{4}\right) . \cos\left(\frac{x}{8}\right) \ldots\ldots \infty = \frac{\sin x}{x}$

taking log on both sides

$\log\left(\cos\frac{x}{2}\right) + \log\left(\cos\frac{x}{4}\right) + \log\left(\cos\frac{x}{8}\right) + \ldots\ldots\ldots = \log(\sin x) - \log x$

Diff w.r.t. x

$\frac{1}{\cos\left(\frac{x}{2}\right)} . \left(-\sin\frac{x}{2}\right) . \left(\frac{1}{2}\right) + \frac{1}{\cos\frac{x}{4}} . \left(-\sin\left(\frac{x}{4}\right)\right)\left(\frac{1}{4}\right) + \ldots\ldots = \frac{1}{\sin x} . \cos x - \frac{1}{x}$

$$\Rightarrow -\frac{1}{2}\tan\frac{x}{2} - \frac{1}{4}\tan\frac{x}{4} - \ldots\ldots = \cot x - \frac{1}{x}$$

$$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \ldots\ldots = -\cot x + \frac{1}{x}$$

Diff again wrt. X

$$\frac{1}{2}sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{4}sec^2\left(\frac{x}{4}\right)\left(\frac{1}{4}\right) + \ldots\ldots = cosec^2 x - \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{2^2}.sec^2\left(\frac{x}{2}\right) + \frac{1}{2^4}.sec^2\left(\frac{x}{4}\right) - 1\ldots\ldots = cosec^2 x - \frac{1}{x^2}$$ Ans.

Q5. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)} + 1.$

Show that $\frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}$

Sol.5 We have $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{dx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$

taking LCM

$$y = \frac{ax^2+bx(x-a)+c(x-a)(x-b)+(x-a)(x-b)(x-c)}{(x-a)(x-b)(x-c)}$$

(open all the brackets in N^r by yourself)

we get

$$\frac{dy}{dx} = \frac{x^3}{(x-a)(x-b)(x-c)}$$

taking log on both sides

$$\log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$$

$\ldots\ldots\{using \log properties\}$

Diff w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = \frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x} + \frac{1}{x} + \frac{1}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{x-a-x}{x(x-a)} + \frac{x-b-x}{x(x-b)} + \frac{x-c-x}{x(x-c)}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{-a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}\right]$$

$$= \frac{y}{x}\left[\frac{-a}{x-a} - \frac{b}{x-b} - \frac{c}{x-c}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right]$$ Ans.

Parametric Functions

Q6. If $x = asec^3\theta$ and $y = atan^3\theta$. Find $\frac{dy}{dx}$ at $\theta = \pi/3$.

Sol.6 We have

$x = asec^3\theta$

Diff w.r.tθ

$\frac{dx}{d\theta} = a.3sec^e\theta.sec\theta.\tan\theta$

$\frac{dx}{d\theta}3asec^3\theta\tan\theta$

we have $y = a\tan^3\theta$

Diff w.r.t θ

$\frac{dx}{d\theta} = 3a\tan^2\theta.sec^2\theta$

Now $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$\frac{dy}{dx} = \frac{3a\tan^2\theta sec^2\theta}{3asec^3\theta\tan\theta}$

$\frac{dy}{dx} = \frac{\tan\theta}{sec\theta}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

Now $\left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \sin(\pi/3) = \frac{\sqrt{3}}{2}$

Ans.

Q7. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$. Find $\frac{dy}{dx}$ at $\theta = \pi/6$.

Sol.7 We have , $x = a(\cos\theta + \theta\sin\theta)$

Diff w.r.t. θ

$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta)$

$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta$

we have,$y = a(\sin\theta - \theta\cos\theta)$

Diff w.r.t θ

$\frac{dy}{d\theta} = a(\cos\theta - (-\theta\sin\theta + \cos\theta))$

$\Rightarrow \frac{dy}{d\theta}a(\cos\theta + \theta\sin\theta - \cos\theta)$

$\Rightarrow \frac{dy}{dx} = a\theta\sin\theta$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta}$

$\Rightarrow \frac{dy}{dx} = \tan\theta$

$\left(\frac{dy}{dx}\right)_{\theta=\pi/6} = \tan(\pi/6) = 1/\sqrt{3}$ Ans.

Q8. If $x = \cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right)$ and $y = \sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$. Find $\frac{dy}{dx}$.

Sol.8 We have , $x = \cos^{-1}\left(\frac{1}{\sqrt{1+t^2}}\right)$

$x = \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) = \cos^{-1}\left(\frac{1}{sec\theta}\right)$

$\Rightarrow$ $x = \cos^{-1}(\cos\theta)$

$\Rightarrow$ $x = \theta$

$\Rightarrow$ $x = \tan^{-1}t$ {replacingθ}

Diff w.r.t. 't'

$\frac{dx}{dt} = \frac{1}{1+t^2}$

Now $y = \sin^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

put $t = \tan\theta$

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)$$

$\Rightarrow$ $y = \sin^{-1}\left(\frac{\tan\theta}{sec\theta}\right) = \sin^{-1}\left(\frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right)$

$\Rightarrow$ $y = \sin^{-1}(\sin\theta)$

$\Rightarrow$ $y = \theta$

$\Rightarrow$ $y = \tan^{-1}t$ .....{replacingθ}

Diff w.r.t 't'

$$\frac{dy}{dt} = \frac{1}{1+t^2}$$

Now , $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{1+t^2}}{\frac{1}{1+t^2}} = 1$ Ans.

Q9. If $x = a\left(\cos t + \log\left(\tan\frac{t}{2}\right)\right)$ and $y = a\sin t$

Sol.9 We have $x = a\left(\cos t + \log\left(\tan\frac{t}{2}\right)\right)$

Diff w.r.t t

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}.sec^2\left(\frac{t}{2}\right).\left(\frac{1}{2}\right)\right)$$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{\frac{1}{\cos^2(t/2)}}{\frac{\sin(t/2)}{\cos(t/2)}}.\frac{1}{2}\right]$$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right).\cos\frac{t}{2}}\right]$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}.sec^2\left(\frac{t}{2}\right).\left(\frac{1}{2}\right)\right) \quad .....\left\{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2} = \sin\theta\right\}$$

$$\frac{dx}{dt} = a\left(\frac{-\sin^2 t+1}{\sin t}\right)$$

$\frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right)$ ....$\{1 - \sin^2\theta = \cos^2\theta\}$

Now, y = a sin t

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{a\cos^2 t}{\sin t}}{a\cos t} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t} = \tan t$$

$\therefore$ $\frac{dy}{dx} = \tan t$ Ans.

Q10. If $x = \sqrt{a^{\sin-1t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$. Show that $\frac{dy}{dx} = \frac{-y}{x}$.

Sol.10 We have, $x = \sqrt{a^{\sin^{-1}t}}$

$$x = (a^{\sin^{-1}t})^{1/2}$$

taking log on both sides

$$\log y = \frac{1}{2}\log(a^{\cos^{-1}t})$$

$\Rightarrow$ $\log y =$

$$\log x = \frac{1}{2}\log(a^{\sin^{-1}t})$$

$\Rightarrow \quad \log x = \frac{1}{2}.\sin^{-1}t.\log a$

$\ldots.\{\log m^n = n\log m\}$

Diff w.r.t t

$$\frac{1}{x}.\frac{dx}{dt} = \frac{1}{2}.\log a.\frac{1}{\sqrt{1-t^2}}$$

$\Rightarrow \quad \frac{dx}{dt} = \frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$

$\Rightarrow \quad y = \sqrt{a^{\cos-1t}}$

taking log on both sides

$\frac{1}{2}\cos^{-1}t.\log a \ldots\{\log m^n = n\log m\}$

Diff w.r.t 't'

$$\frac{1}{y}.\frac{dy}{dt} = \frac{1}{2}\log a\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$\Rightarrow \quad \frac{dy}{dt} = \frac{-y}{2}\log a.\frac{1}{\sqrt{1-t^2}}$

Now $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{y}{2}\log a.\frac{1}{\sqrt{1-t^2}}}{\frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}}$

$\frac{dy}{dx} = \frac{-y}{x}$ Ans.