

Assignment of Continuity and Differentiation

1. Prove that $y = |x|$ is continuous at $x=0$, but not differentiable at $x=0$.
2. Prove that $y = [x]$ is continuous at $x=0$.
3. $F(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ x, & \text{when } x = 0 \end{cases}$, Check the differentiability and continuity at $x = 0$.
4. Does $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ exists?
5. Find the derivative of the following functions w.r.t. x:
 - (i) $\tan^{-1}(1 + x^2)$
 - (ii) Sec $\tan^{-1}\sqrt{x}$
 - (iii) $\sqrt{\sin\sqrt{x}}$
 - (iv) $\sqrt{x}\sin x$
 - (v) $\tan x^0$
 - (vi) $\cos\left(\frac{1-x^2}{1+x^2}\right)$
 - (vii) $\sqrt{x}\sin x + \sin\sqrt{x}$
 - (viii) $\tan\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$
 - (ix) $\sin\sqrt{\cos\sqrt{\tan mx}}$
 - (x) $\sin[\cos\{\tan(\cot x)\}]$
 - (xi) $\cot^{-1}\frac{1-x}{1+x}$
 - (xiii) $\tan^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$
 - (xiv) $\tan^{-1}\left[\frac{4\sqrt{x}}{1-4x}\right]$
 - (xv) $\sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$
 - (xvi) $\sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$
 - (xvii) $x^{(x^x)}$
 - (xviii) $(x^x)^x$
 - (Xix) $\log_7 \log_7 x$
 - (xx) $\left(\frac{3+x}{1+x}\right)^{2+3x}$
 - (xxi) $x^{\cos x} + (\sin x)^{\tan x}$
 - (xxii) $(\log x)^x + x^{\log x} + (\log x)^{\log x}$.
6. If $y = \cos^{-1}x$, Find $\frac{d^2y}{dx^2}$ in term of y alone.
7. If $y = a \cos(\log x) + b \sin(\log x)$, Show that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + Y = 0$.
8. If $y = x^x$, show that $\frac{d^2y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$.
9. If $(x-a)^2 + (y-b)^2 = c^2$, $c > 0$. Prove that $\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}^{\frac{3}{2}}$ is a constant independent of a & b.
10. If $x = a\{\cos t + t \sin t\}$, $y = a\{\sin t - t \cos t\}$, Prove that $\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$.
11. Verify Rolle's theorem for $f(x) = x^3(x-1)^2$, $0 \leq x \leq 1$.
12. Verify L.M.V theorem for the function $f(x) = x(x-1)(x-2)$ in the interval $(0, \frac{1}{2})$.
13. Prove that $g(x) = x - [x]$ is discontinuous at all integral parts.
14. If $f(x) = \begin{cases} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, Check the continuity at $x = 0$.
15. Show that the $\cot x$ is a continuous function in its domain.
16. Show that $f(x) = |1 - x + |x||$ is a continuous function.

17. Find all the points of discontinuity for greatest integer function.

18. If the function $f(x) = \begin{cases} 3ax + b & , x > 1 \\ 11 & , x = 1 \\ 5ax - 2b & , x < 1 \end{cases}$, is continuous at $x = 1$, find the values of a & b.

19. $F(x) = \begin{cases} \frac{1-\cos 4x}{px^2} & , x \neq 0 \\ k & , x = 1 \end{cases}$, is continuous at $x = 0$, find the value of k.

20. If the function $f(x) = \begin{cases} \frac{1-\sin^2 x}{3\cos^2 x} & , x < \frac{\pi}{2} \\ a & , x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & , x > \frac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$, find the values of a & b.

21. Prove that $\frac{d}{dx} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right] = \sqrt{a^2 - x^2}$.

22. Prove that: $\frac{d}{dx} \left[\cot^{-1} \sqrt{\frac{1-\sin x}{1+\sin x}} \right] = \frac{1}{2}$.

23. Differentiate: $\tan^{-1} \frac{2x}{1-x^2}$ w.r.t. $\sin^{-1} \frac{2x}{1+x^2}$.

24. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = -(1-x)^{-2}$.

25. If $y = \log \tan(\frac{\pi}{4} + \frac{x}{2})$, show that $\frac{dy}{dx} - \sec x = 0$.

26. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

27. If $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$. find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{2}$.

28. If $y = a^{t+\frac{1}{t}}$, $x = \left(t + \frac{1}{t}\right)^a$, find $\frac{dy}{dx}$.

29. If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$. prove that $\frac{dy}{dx} = \frac{-y}{x}$.

30. If $y = x - x^2$, Find derivative of y^2 w.r.t. x^2 .

31. If $y = \begin{vmatrix} a & b & c \\ f(x) & g(x) & h(x) \\ l & m & n \end{vmatrix}$. Prove that $\frac{dy}{dx} = \begin{vmatrix} a & b & c \\ f'(x) & g'(x) & h'(x) \\ l & m & n \end{vmatrix}$.

32. If $f(x) = |x|^3$. Show that $f''(x)$ exists for all real x and find it.

33. If $x = f(t)$, $y = g(t)$. Prove that $\frac{d^2y}{dx^2} = \frac{\frac{(dx)}{dt} \left(\frac{d^2y}{dt^2} \right) - \left(\frac{d^2x}{dt^2} \right) \left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)^3}$.

34. find differential coefficient of $\log_{10} x$ w.r.t. $\log_x 10$.

35. IF $\sqrt{X+Y} + \sqrt{Y-X} = C$. Prove that $\frac{d^2y}{dx^2} = \frac{2}{c^2}$.