

## Applications of Derivatives

### Key points to remember

- Rate of change-** Let  $y = f(x)$  be a function then the rate of change of  $y$  with respect to  $x$  is given by  $\frac{dy}{dx} = f'(x)$  where a quantity  $y$  varies with another quantity  $x$ .  
 $\frac{dy}{dx} \bigg|_{x=x_0}$  or  $f'(x_0)$  represents the rate of change of  $y$  w.r.t.  $x$  at  $x = x_0$ .
- If  $x = f(t)$  and  $y = g(t)$   
 By the chain rule  
 $\frac{dy}{dx} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$  if  $\frac{dx}{dt} \neq 0$
- (i) A function  $f(x)$  is said to be **increasing** on an interval  $(a, b)$  if  $x_1 < x_2$  in  $(a, b)$  this implies  $f(x_1) \leq f(x_2) \forall x_1, x_2 \in (a, b)$ . Alternatively if  $f'(x) \geq 0 \forall x \in (a, b)$ , then  $f(x)$  is increasing function in  $(a, b)$ .
- (i) A function  $f(x)$  is said to be **decreasing** on an interval  $(a, b)$  if  $x_1 \leq x_2$  in  $(a, b)$  this implies  $f(x_1) \geq f(x_2) \forall x_1, x_2 \in (a, b)$ . Alternatively if  $f'(x) \leq 0 \forall x \in (a, b)$ , then  $f(x)$  is decreasing function in  $(a, b)$ .
- The **equation of tangent** at the point  $(x_0, y_0)$  to a curve  $y = f(x)$  is given by  
 $y - y_0 = \frac{dy}{dx}(x - x_0)$
- Where  $\frac{dy}{dx}$  is **slope of tangent** at  $(x_0, y_0)$
- The **equation of normal** at the point  $(x_0, y_0)$  to a curve  $y = f(x)$  is given by  
 $y - y_0 = \frac{-1}{\frac{dy}{dx}}(x - x_0)$
- If  $\frac{dy}{dx} = 0$ , then the tangent is parallel to  $x$ -axis at  $(x_0, y_0)$ , then the equation of tangent is  $x = x_0$ .
- If  $\frac{dy}{dx}$  does not exist then the normal is parallel to  $x$ -axis at  $(x_0, y_0)$ , then the equation of normal is  $y = y_0$ .
- Let  $f$  be a function. Let point  $c$  be in the domain of the function  $f$  at which either  $f'(x) = 0$  or  $f$  is not derivable is called a **critical point** of  $f$ .
- First Derivative Test** : Let  $f$  be a function defined on an open interval  $I$ . Let  $f$  be continuous at a critical point  $c \in I$ . Then  
 If  $f'(x)$  changes sign from positive to negative as we pass through  $c$ , then  $c$  is called **point of local maxima**.  
 If  $f'(x)$  changes sign from negative to positive as we pass through  $c$ , then  $c$  is called **point of local minima**.

## Increasing and Decreasing Functions

### ASSIGNMENT-1

Q 1) Find the intervals in which the following functions are strictly increasing , increasing , strictly decreasing and decreasing.

(i)  $f(x) = (x+1)^3(x-1)^3$  ]

(ii)  $f(x) = x^3 - 6x^2 + 9x + 15$

(iii)  $f(x) = x^3 - 12x^2 + 36x + 17$

(iv)  $f(x) = -2x^3 - 9x^2 - 12x + 1$

(v)  $f(x) = (x+2)e^{-x}$

Q 2) Find the intervals for which the following functions are strictly increasing and strictly decreasing.

(i)  $f(x) = x/\log x$   $x > 0$  and  $x \neq 1$

(ii)  $f(x) = x/(1+x^2)$

(iii)  $f(x) = 2\log(x-2) - x^2 + 4x + 1$

(iv)  $f(x) = x^x$   $x > 0$

(v)  $f(x) = x/2 + 2/x$   $x \neq 0$

(vi)  $f(x) = (4x^2 + 1)/x$

(vii)  $f(x) = x^4 - x^3/3$

(viii)  $f(x) = (x-2)/(x+1)$ ,  $x \neq -1$

Q 3) Show that the function  $f(x) = x^3 - 6x^2 + 12x - 18$  is an increasing function on  $\mathbf{R}$ .

Q 4) Find the intervals in which the function  $f(x) = \sin x - \cos x$ ,  $0 < x < 2\pi$  is strictly increasing or decreasing.

Q 5) (i) Show that  $f(x) = e^{1/x}$  is a decreasing function for  $x \neq 0$

(ii) Show that  $f(x) = \log_a x$   $0 < a < 1$  is a decreasing function for all  $x > 0$ .

(iii) Show that  $f(x) = x - \sin x$  is an increasing function for all  $x \in \mathbf{R}$ .

(iv) Show that  $f(x) = \cos^2 x$  is a decreasing function on  $(0, \pi/2)$ .

(v) Show that  $f(x) = \sin x$  is an increasing function on  $(-\pi/2, \pi/2)$ .

(vi) Show that  $f(x) = \sin(2x + \pi/4)$  is a decreasing function on  $(3\pi/8, 5\pi/8)$ .

Q 6) Find the values of  $k$  for which  $f(x) = kx^3 - 9kx^2 + 9x + 3$  is increasing on  $\mathbf{R}$

Q 7) Show that  $f(x) = (3/x) + 7$  is a decreasing function for  $x \in \mathbf{R}$  ( $x \neq 0$ ).

Q 8) Without using derivatives, Prove that

- (i)  $f(x) = \log_e x$  is increasing on  $(0, \infty)$
- (ii)  $f(x) = ax + b$ , where  $a > 0$  &  $a$  and  $b$  are constants is an increasing function.
- (iii)  $f(x) = ax + b$ , where  $a < 0$  &  $a$  and  $b$  are constants is a decreasing function.
- (iv)  $f(x) = -3x + 12$  is strictly decreasing on  $\mathbb{R}$ .
- (v)  $f(x) = x^2$  is strictly increasing on  $[0, \infty)$  & decreasing on  $(-\infty, 0]$ .

### ANSWER KEY

1. ii. strictly increasing in  $(-\infty, -3) \cup (1, \infty)$  and strictly decreasing in  $(-3, 1)$   
 increasing in  $(-\infty, -3] \cup [1, \infty)$  and decreasing in  $[-3, 1]$
- iii. strictly increasing in  $(-\infty, 2) \cup (6, \infty)$  and strictly decreasing in  $(2, 6)$   
 increasing in  $(-\infty, 2] \cup [6, \infty)$  and decreasing in  $[2, 6]$
- iv. strictly increasing in  $(-2, -1)$  and strictly decreasing in  $(-\infty, -2) \cup (-1, \infty)$   
 increasing in  $[-2, -1]$  and decreasing in  $(-\infty, -2] \cup [-1, \infty)$
- v. strictly increasing in  $(-\infty, -1)$  and strictly decreasing in  $(-1, \infty)$   
 increasing in  $(-\infty, -1]$  and strictly decreasing in  $[-1, \infty)$
2. i. strictly increasing in  $(e, \infty)$  and strictly decreasing in  $(0, e) - \{1\}$
- ii. strictly increasing in  $(-1, 1)$  and strictly decreasing in  $(-\infty, -1) \cup (1, \infty)$
- iii. strictly increasing in  $(2, 3)$  and strictly decreasing in  $(3, \infty)$
- iv. strictly increasing in  $(1/e, \infty)$  and strictly decreasing in  $(0, 1/e)$
- v. strictly increasing in  $(-\infty, -2) \cup (2, \infty)$  and strictly decreasing in  $(-2, 0) \cup (0, 2)$
- vi. strictly increasing in  $(-\infty, -1/2) \cup (1/2, \infty)$  and strictly decreasing in  $(-1/2, 0) \cup (0, 1/2)$
- vii. strictly increasing in  $(1/4, \infty)$  and strictly decreasing in  $(-\infty, 1/4)$
- viii. strictly increasing in  $\mathbb{R} - \{-1\}$
4. increasing on  $(0, \pi/4) \cup (\frac{5\pi}{4}, 2\pi)$  and decreasing on  $(\frac{\pi}{4}, \frac{5\pi}{4})$
6.  $k \in (0, 1/3)$

## Assignment

## Tangent and Normal

- Find the values of a and b if the slope of the tangent to the curve  $xy + ax + by = 2$  at (1,1) is 2.
- If tangent to the curve  $y = x^3 + ax + b$  at (1,-6) is parallel to the line  $x - y + 5 = 0$ . Find a and b.
- Find a point on the curve  $y = x^3 - 3x$  where tangent is parallel to a chord joining (1,-2) and (2,2).
- Find the points on the curve  $y^2 = 2x^3$  at which the tangent lines are inclined at an angle of  $45^\circ$  with the x- axis.
- Find the points on the curve  $y = x^2$  where slope of the tangent is equal to x coordinate of the point.
- Find the points on the curve  $y = x^2$  at which the tangents are equally inclined with the axes.
- Find the point on the curve  $y = 3x^2 + 4$  at which tangent is perpendicular to the line with the slope  $-1/6$ .
- find the points on the curve  $\frac{x^2}{9} + \frac{y^2}{16} = 1$  at which tangents are  
(1)parallel to x- axis.  
(2)parallel to y- axis.
- Show that the line  $\frac{x}{a} + \frac{y}{b} = 1$  touches the curve  $y = be^{\frac{-x}{a}}$  at the point where it crosses the y- axis.
- For the curve  $y = 4x^3 - 2x^5$ . Find all points at which the tangent passes through the origin.
- Show that the normal at any point  $\alpha$  to the curve  $x = a(\cos \alpha + a \sin \alpha)$  and  $y = a(\sin \alpha - a \cos \alpha)$  is at a constant distance from the origin.
- Find equation of tangent to the curve  $x = a \sin \alpha, y = 1 + \cos \alpha$  at  $\alpha = \frac{\pi}{4}$ .
- Find equation of normal to the curve  $x^2 + 2y^2 - 4x - 6y + 8 = 0$  at the point whose abscissa is 2.
- Show that the condition that the curves  $ax^2 + by^2 = 1$  and  $cx^2 + dy^2 = 1$  should intersect orthogonally is that  $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$ .
- Show that the curves  $4x = y^2$  and  $4xy = k$  cut at right angles if  $k^2 = 512$ .

## Answer Key

1.  $a=5, b=-4$     2.  $a=-2, b=-5$     3.  $\pm \sqrt{\frac{7}{3}}, \mp \frac{2}{3} \sqrt{\frac{7}{3}}$     5. (0,0) .

7. (1,7)    8(i) (0,4)(0,-4) (ii) (3,0)(-3,0)    10. (0,0),(1,2),(-1,-2)    13.  $x = 2$

## Assignment

## Maxima - Minima

1. Show that all the rectangles with a given perimeter, the square has the largest area.
2. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
3. Find the volume of the largest cylinder that can be inscribed in a sphere of radius  $r$  cm.
4. Show that a cylinder of given volume which is open at the top has minimum total surface area, provided its height is equal to the radius of its base.
5. Show that the height of the closed cylinder of given surface and maximum volume is equal to the diameter of its base.
6. A jet of enemy is flying along the curve  $y = x^2 + 2$ . A soldier is placed at the point the point  $(3, 2)$ . What is the nearest distance between the soldier and the jet?
7. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when depth of the tank is half of its width.
8. Show that among all positive numbers  $x$  and  $y$  with  $x^2 + y^2 = r^2$ , the sum  $x + y$  is largest when  $x = y = \frac{r}{\sqrt{2}}$ .
9. A wire of length 20m is to be cut into 2 pieces. One of the piece will be bent into the shape of a square and the other into the shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square is minimum.
10. A large window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, Find the dimensions of the rectangle that will produce the largest area of the window.
11. Determine the points on the curve  $x^2 = 4y$  which are nearest to the point  $(0, 5)$ .
12. Determine the point on the curve  $y^2 = 4x$  which is nearest to the point  $(2, -8)$ .
13. Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.
14. Divide 64 into two parts such that the sum of the cubes of two parts is minimum.
15. How should we choose two numbers, each greater than or equal to  $-2$ , whose sum is  $1/2$  and sum of the first and cube of the second is minimum.
16. Divide 15 into two parts such the square of one multiplied with cube of the other is minimum.
17. A wire of 25m is to be cut into 2 pieces. One of the pieces is to be made into a square and the other into the circle. What should be lengths of the two pieces so that combined area of the square and the circle is minimum.
18. A square piece of tin of side 18 cm is to be made into box without top cutting a square from each corner and folding up the flaps to form a box. What should be side of the

square to be cut off so that the volume of the box is maximum? Also find the maximum volume.

19. The space  $s$  described in time  $t$  by a particle moving in a straight line is given by  $s = t^5 - 40t^3 + 30t^2 + 80t - 250$ . Find the minimum value of the acceleration.
20. Show that cone of the greatest volume that which can be inscribed in a given sphere has an altitude equal to  $\frac{2}{3}$  of the diameter of the sphere.
21. If  $f(x) = x^3 + ax^2 + bx + c$  has maximum at  $x = -1$  and minimum at  $x = 3$ . Determine  $a, b, c$ .
22. A closed cylinder has volume  $2156 \text{ cm}^3$ . What will be the radius of its base so that its total surface area is minimum.
23. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius  $5\sqrt{3}$  is  $500\pi \text{ cm}^3$ .
24. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.

#### ANSWER KEY

3.  $\frac{4}{3} \frac{\pi r^3}{\sqrt{3}}$       6.  $\sqrt{5}$       9.  $\frac{20\sqrt{3}}{9+4\sqrt{3}}, \frac{60}{9+4\sqrt{3}}$       10.  $\frac{12}{6-\sqrt{3}}, \frac{24-6\sqrt{3}}{6-\sqrt{3}}$
11.  $(\pm 2\sqrt{3}, 3)$       12.  $(4, -4)$       13.  $\frac{15}{2}, \frac{15}{2}$       14. 32, 32
15.  $(\frac{1}{2}, -\frac{1}{\sqrt{3}}), \frac{1}{\sqrt{3}}$       16. 6, 9      17.  $\frac{100}{\pi+4}, \frac{25\pi}{\pi+4}$       18. 3cm, 432cm<sup>3</sup>
19.  $a = -260$  at  $t = 2$       21.  $a = -3, b = -9, c \in \mathbb{R}$       22. 7cm.