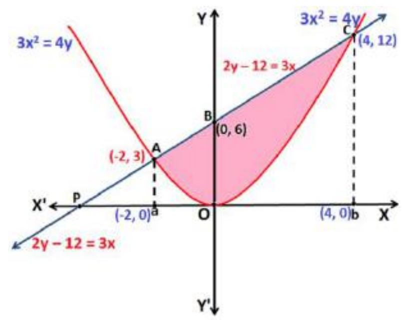
APPLICATION OF INTEGRALS
Class 12th

Q.1) Find the area bounded by the curves $4y = 3x^2$ and $2y = 3x + 12$.

Sol.1) 1) $3x^2 = 4y$
(.) parabola
(.) vertex (0,0)
(.) open towards $+ve$ y-axis



2) $2y = 3x + 12$
(.) line
(.) points (0, 6) and (–4, 0)

Intersection points:
Solving $3x^2 = 4y$ and $2y = 3x + 12$
We have $(4, 12)$ and $(-2, 3)$

Required area = $\int_{-2}^{4}\left(\frac{3x+12}{2}\right) - \left(\frac{3x^2}{4}\right) dx$

$= \frac{1}{4}\int_{-2}^{4}(6x + 24 - 3x^2)\, dx$

$= \frac{3}{4}\int_{-2}^{4}(2x + 8 - x^2)\, dx$

$= \frac{3}{4}\left[x^2 + 8x - \frac{x^3}{3}\right]_{-2}^{4}$

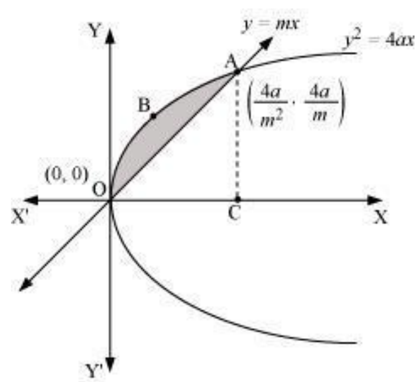
$= \frac{3}{4}\left[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right]$

$= \frac{3}{4}\left[\frac{80}{3} + \frac{28}{3}\right] = \frac{3}{4}\left(\frac{108}{3}\right) = \frac{108}{4} = 27$

∴ Required area = 27 square units ans.

Q.2) Find the area bounded by $y^2 = 4ax$ and $y = mx$.

Sol.2) 1) $y^2 = 4ax$
(.) parabola
(.) vertex (0,0)
(.) open towards $+ve$ x-axis



2) $y = mx$
(.) line
(.) passing through (0,0)

Intersection Points:
Solving $y^2 = 4ax$ and $y = mx$
we have, (0,0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$

Required area = $\int_{0}^{\frac{4a}{m^2}}\left(2\sqrt{a}\sqrt{x} - mx\right) dx$

$= \int_{0}^{\frac{4a}{m^2}}\left(2\sqrt{a}\,.\frac{2}{3}x^{\frac{3}{2}} - \frac{mx^2}{2}\right) dx$

$= \left[\frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{\frac{3}{2}} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2\right] - [0]$

$= \frac{4\sqrt{a}}{3}\,.\left(\frac{8a\sqrt{a}}{m^3}\right) - \frac{m}{2}\left(\frac{16a^2}{m^4}\right)$

$= \frac{32a^2}{3m^3} - \frac{8a^2}{m^3}$

$= \frac{8a^2}{m^3}$

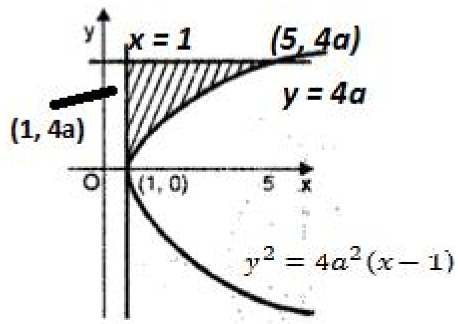
$\therefore$ Required area $\frac{8a^2}{m^3}$ square units. ans.

Q.3) Find the area bounded by curve $y^2 = 4a^2(x-1)$ and lines $x = 1$ and $y = 4a$.

Sol.3)

1) $x = 1$

(.) line parallel to y-axis at $(1, 0)$



2) $y = 4a$

(.) line parallel to x-axis at $(0, 4a)$

3) $y^2 = 4a^2(x-1)$

(.) shifting parabola

(.) vertex (1,0)

(.) open towards $+ve$ x-axis

Intersection Points:

Solving $y^2 = 4a^2(x-1)$ and $y = 4a$

We have $16a^2 = 4a^2(x-1)$

$4 = x - 1 \Rightarrow x = 5 \quad \therefore$ point $(5, 4a)$

Required area = $\int_1^5 \left(4a - 2a\sqrt{x-1}\right) dx$

$= \left[4ax - 2a \cdot \frac{2}{3}(x-1)^{\frac{3}{2}}\right]_1^5$

$= \left[20a - \frac{4a}{3}(4)^{\frac{3}{2}}\right] - [4a - 0]$

$= 20a - \frac{32a}{3} - 4a$

$= 16a - \frac{32a}{3}$

$= \frac{16a}{3}$

$\therefore$ Required area = $\frac{16a}{3}$ sq. units. ans.

Q.4) Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and y–axis.

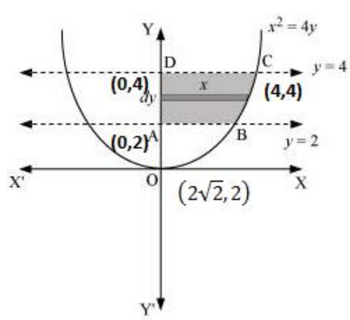
Sol.4)

1) $x^2 = 4y$

(.) parabola

(.) vertex (0,0)

(.) open towards $+ve$ y-axis



2) $y = 2$

(.) line parallel to x-axis at (0,2)

3) $y = 4$

(.) line parallel to y-axis at (0,4)

4) y-axis

Intersection points:

Solving $x^2 = 4y$ and $y = 4$ we have (4,4)

Solving $x^2 = 4y$ and $y = 2$ we have $(2\sqrt{2}, 2)$

Required area = $\int_0^{2\sqrt{2}}(4-2)\,dx+\int_{2\sqrt{2}}^{4}\left(4-\frac{x^2}{4}\right)dx$

$= (2x)_0^{2\sqrt{2}}+\left(4x-\frac{x^3}{12}\right)_{2\sqrt{2}}^{4}$

$= 4\sqrt{2}+\left[16-\frac{64}{12}\right]-\left[8\sqrt{2}-\frac{16\sqrt{2}}{12}\right]$

$= 4\sqrt{2}+\frac{128}{12}-\frac{80\sqrt{2}}{12}$

$= \frac{128-32\sqrt{2}}{12}$

$= \frac{32-8\sqrt{2}}{3}$

$\therefore$ Required area = $\frac{32-8\sqrt{2}}{3}$ sq. units

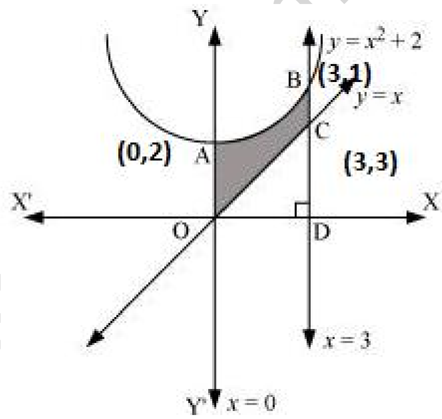
Q.5) Find the area of the region bounded by curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

Sol.5) 1) $y = x^2 + 2 \Rightarrow x^2 = y - 2$

(.) shifting

(.) vertex $(0,2)$

(.) open towards $+ve$ y-axis



2) $y = x$

(.) line passing through $(0,0)$

3) $x = 0$

(.) y-axis

4) $x = 3$

(.) line parallel to y-axis at $(3,0)$

Intersection point:

Solving $y = x^2 + 2$ and $x = 3$

We have $x = 3$ & $y = 1$

$\therefore$ $(3,1)$

Required area = $\int_0^3 (x^2+2-x)\,dx$

$= \left(\frac{x^3}{3}+2x-\frac{x^2}{2}\right)_0^3$

$= \left(9+6-\frac{9}{2}\right)-0$

$= \frac{21}{2}$

$\therefore$ Required area = $\frac{21}{2}$ square unit. ans.

Q.6) Find the area of the region $0 \le y \le x^2+1$; $0 \le y \le x+1$; $0 \le x \le 2$

Sol.6) 1) $y \ge 0$

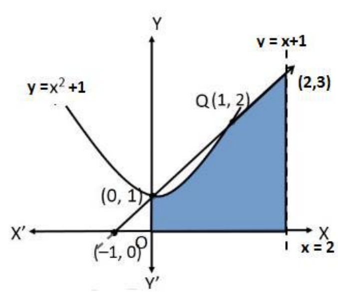
2) $y \le x^2+1 \quad \Rightarrow x^2 \ge y - 1$

(.) shifting parabola

(.) vertex $(0,1)$

(.) open towards $+ve$ y-axis

(.) solution outside the parabola



3) $y \le x + 1$

(.) line passing through $(0,1)$ & $(-1,0)$
4) (.) solution towards the origin
5) $x \geq 0$
$x \leq 2$
(.) line parallel to y-axis at $(2,0)$
(.) solution towards the origin
$x \geq 0$ & $y \geq 0$ means solution in 1st quadrant.

Intersection points

Solving $y = x^2 + 1$ and $y = x + 1$
We have $(0,1)$ and $(1,2)$

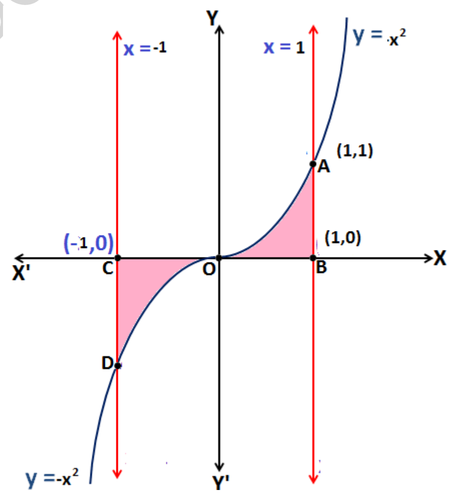
Required area = $\int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx$

$$= \left(\frac{x^3}{3} + x\right)_0^1 + \left(\frac{x^2}{2} + x\right)_1^2$$
$$= \left[\frac{1}{3} + 1\right] + \left[(2+2) - \left(\frac{1}{2} + 1\right)\right]$$
$$= \frac{4}{3} + 4 - \frac{3}{2}$$
$$= \frac{8+24-9}{6} = \frac{23}{6}$$

$\therefore$ Required area = $\frac{23}{6}$ square units ans.

Q.7) Find the area bounded by the curves $y = |x|$, x-axis, $x = -1$ and $x = 1$.

Sol.7) 1) $y = x|x| \longrightarrow$ two parabolas
(.) $y = x^2$; $x \geq 0$
vertex (0,0) open towards $+ve$ y-axis
(.) $y = -x^2$; $x < 0$
Vertex $(0,0)$ open towards $-ve$ y-axis
2) x-axis
3) $x = 1 \longrightarrow$ line parallel to y-axis at $(1,0)$
4) $x = -1 \longrightarrow$ line parallel to y-axis at $(-1,0)$



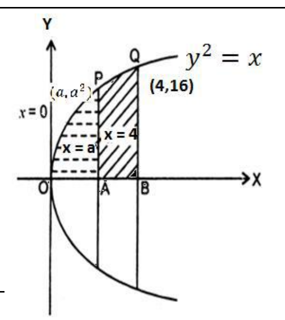
Required area = $\int_{-1}^{0} -(-x)^2\,dx + \int_0^1 x^2\,dx$

$$= \left(\frac{x^3}{3}\right)_{-1}^{0} + \left(\frac{x^3}{3}\right)_0^1$$
$$= \left(0 - \left(-\frac{1}{3}\right)\right) + \left(\frac{1}{3} - 0\right)$$
$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Required area = $\frac{2}{3}$ sq. units ans.

Q.8) The area between $x = y^2$ and $x = 4$ is divided in to two equal parts by line $x = a$. Find value of a.

Sol.8) 1) $y^2 = x$
(.) parabola, vertex $(0,0)$
open towards $+ve$ x-axis
2) $x = 4$
(.) line parallel to y-axis at $(4,0)$
3) $x = a$
(.) line parallel to y-axis at $(a, 0)$
Area of region A:



$= 2\int_0^a (\sqrt{x} - 0)\, dx$ ....{due to symmetry}

$= 2 . \frac{2}{3}\left(x^{\frac{3}{2}}\right)_0^a$

$= \frac{4}{3} a\sqrt{a}$ sq. units

Area of region B:

$= 2\int_0^4 (\sqrt{x} - 0)\, dx$ ....{due to symmetry}

$= 2 . \frac{2}{3}\left(x^{\frac{3}{2}}\right)_a^4$

$= \frac{4}{3}\left[8 - a^{\frac{3}{2}}\right]$ sq. units

We are given that, area of region A = area of region B

$\Rightarrow \quad \frac{4}{3} a\sqrt{a} = \frac{4}{3}\left(8 - a^{\frac{3}{2}}\right)$

$\Rightarrow \quad a^{\frac{3}{2}} = 8 - a^{\frac{3}{2}}$

$\Rightarrow \quad 2a^{\frac{3}{2}} = 8$

$\Rightarrow \quad a^{\frac{3}{2}} = 4$

$\Rightarrow \quad a = 4^{\frac{2}{3}}$ ans.

Q.9) Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by lines $x = 0, y = 4, x = 0$ and $x = 4$ in to three equal parts.

Sol.9)

1) $y^2 = 4x$

(.) parabola, vertex $(0,0)$, open towards $+ve\ x$-axis

2) $x^2 = 4y$

(.) parabola, vertex $(0,0)$, open towards $+ve\ y$-axis

3) $x = 0$

(.) y-axis

4) $x = 4$

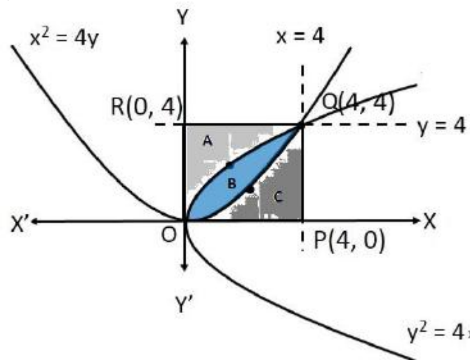
(.) live parallel to y-axis at $(4,0)$

5) $y = 0$

(.) equation of x-axis

6) $y = 4$

(.) line parallel to x-axis at (0,4)



Area of region A

$= \int_0^4 (4 - 2\sqrt{x})\, dx$

$= \left(4x - \frac{4}{3}x^{\frac{3}{2}}\right)_0^4$

$= \left(16 - \frac{4}{3}(8)\right) - (0)$

$= \frac{16}{3}$ sq. units

Area of region B:

$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$

$= \left(\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right)_0^4$

$= \left(\frac{4}{3}(8) - \frac{64}{12}\right) - 0$

	$= \frac{32}{3} - \frac{16}{3}$ $= \frac{16}{3}$ sq. units Area of region C $= \int_0^4 \left(\frac{x^2}{4} - 0\right) dx$ $= \left(\frac{x^3}{12}\right)_0^4$ $= \frac{64}{12} = \frac{16}{3}$ sq. units Clearly, the parabolas divide the area of the square in to three equal parts.
Q.10)	Find the area of the region $\{(x, y): x^2 \leq y \leq \lvert x \rvert\}$
Sol.10) 1) 2)	$x^2 \leq y$ (.) parabola (.) vertex $(0,0)$ (.) open towards $+ve$ y-axis $x^2 = 1$ (.) solution inside the parabola $y \leq \lvert x \rvert$ (.) $y \leq x$; $x \geq 0$ [line passes through (0,0)] (.) $y \leq -x$; $x < 0$ [line passes through (0,0)] Required area = $2\int_0^1 (x - x^2)\, dx$ ....{due to symmetry} $= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$ $= 2\left(\frac{1}{2} - \frac{1}{3}\right)$ $= 2\left(\frac{1}{6}\right) = \frac{1}{3}$ sq. units ans.