APPLICATION OF INTEGRAL

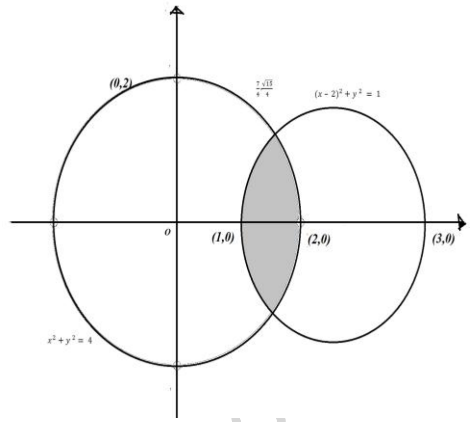
Q.1) Find the area bounded by the curves $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 1$.

Sol.1) 1) $x^2 + y^2 = 4$

(.) circle , center $(0,0)$, rod $= 2$

2) $(x-2)^2 + y^2 = 1$

(.) circle , center $(2,0)$ and rod $= 1$



Intersection point:

Put $y^2 = 4 - x^2$ in $(x-2)^2 + y^2 = 1$

$\Rightarrow (x-2)^2 + 4 - x^2 = 1$

$\Rightarrow x^2 + 4 - 4x + 4 - x^2 = 1$

$\Rightarrow 4x = 7$

$\Rightarrow x = \frac{7}{4}$

$\therefore y^2 = 4 - \frac{49}{16} \quad \Rightarrow y^2 = \frac{15}{16} \quad \Rightarrow y = \frac{\sqrt{15}}{4}$

Required area = $2\int_1^{\frac{7}{4}} \sqrt{1-(x-2)^2}\,dx + 2\int_{\frac{7}{4}}^{2} \sqrt{4-x^2}\,dx$

$$= 2\left[\frac{(x-2)}{2}\sqrt{1-(x-2)^2} + \frac{1}{2}\sin^{-1}(x-2)\right]_1^{\frac{7}{4}} + 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\frac{7}{4}}^{2}$$

$$= 2\left[\left(-\frac{1}{8}\cdot\frac{\sqrt{15}}{4} + \frac{1}{2}\sin^{-1}\left(-\frac{1}{4}\right)\right) - \left(0 + \frac{1}{2}\sin^{-1}(-1)\right)\right] + 2\left[\left(0 + 2\sin^{-1}(1)\right) - \left(\frac{7}{8}\cdot\frac{\sqrt{15}}{4} + 2\sin^{-1}\left(\frac{7}{8}\right)\right)\right]$$

$$= 2\left[\frac{-\sqrt{15}}{32} - \frac{1}{2}\sin^{-1}\left(\frac{1}{4}\right) + \frac{1}{2}\cdot\frac{\pi}{2}\right] + 2\left[2\cdot\frac{\pi}{2} - \frac{7\sqrt{15}}{32} - 2\sin^{-1}\left(\frac{7}{8}\right)\right]$$

$$= \frac{-\sqrt{15}}{16} - \sin^{-1}\left(\frac{1}{4}\right) + \frac{\pi}{2} + 2\pi - \frac{7\sqrt{15}}{16} - 4\sin^{-1}\left(\frac{7}{8}\right)$$

$$= \frac{-8\sqrt{15}}{16} + \frac{5\pi}{2} - \sin^{-1}\left(\frac{1}{4}\right) - 4\sin^{-1}\left(\frac{7}{8}\right)$$

$\therefore$ Required area = $\frac{-\sqrt{15}}{2} + \frac{5\pi}{2} - \sin^{-1}\left(\frac{1}{4}\right) - 4\sin^{-1}\left(\frac{7}{8}\right)$ sq. units ans.

Q.2) Find the area of the region $\{(x,y): y^2 \leq 4x \,,\, 4x^2 + 4y^2 \leq 9\}$.

Sol.2) 1) $y^2 \leq 4x$

(.) parabola

(.) vertex $(0,0)$

(.) open $+ve$ x-axis

(.) solution inside the parabola

2) $4x^2 + 4y^2 \leq 9$

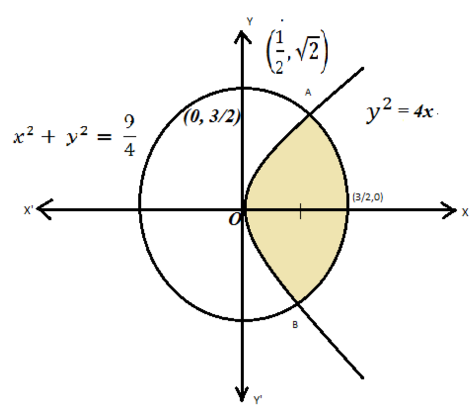
(.) circle

(.) center $(0,0)$

3) $x^2 + y^2 \leq \frac{9}{4}$

(.) radius $= \frac{9}{2}$

(.) solution: Inside the circle



Intersection point

Put $y^2 = 4x$ in $4x^2 + 4y^2 = 9$

$\Rightarrow 4x^2 + 16x - 9 = 0$

$\Rightarrow 4x^2 + 18x - 2x = 0$

$\Rightarrow (2x-1)(2x+9) = 0$

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$x = \frac{1}{2}$ (or) $x = \frac{-9}{2}$ (rejected)

$\therefore y = \sqrt{2}$

point $\left(\frac{1}{2}, \sqrt{2}\right)$

Required area = $2\int_0^{\frac{1}{2}} 2\sqrt{2}\, dx + 2\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\, dx$

$= 4\left[\frac{2}{3} . x^{\frac{2}{3}}\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$

$= \frac{8}{3}\left[\frac{1}{2\sqrt{2}} - 0\right] + 2\left[\left(0 + \frac{9}{8}\sin^{-1}(1)\right) - \left(\frac{1}{4}\sqrt{2} + \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right)\right]$

$= \frac{4}{3\sqrt{3}} + 2\left[\frac{9}{8} . \frac{\pi}{2} - \frac{\sqrt{2}}{4} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$

$= \frac{4}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

$= \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2} + \frac{9\pi}{8} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

$= \frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$ sq. units ans.

Q.3) Find the area of the circle $x^2 + y^2 \le 16$ and parabola $y^2 \ge 6x$.

Sol.3) 1) $x^2 + y^2 \le 16$

(.) circle

(.) center $(0,0)$

(.) radius $= 4$

(.) solution inside the circle

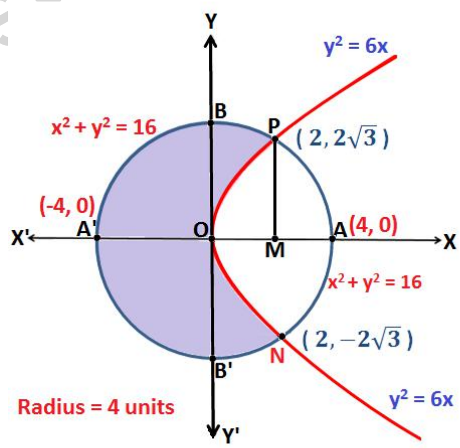
2) $y^2 \ge 6x$

(.) parabola

(.) vertex $(0,0)$

(.) open $+ve$ x-axis

(.) solution outside the parabola



Intersection point

Put $y^2 = 6x$ in $x^2 + y^2 = 16$

$\Rightarrow x^2 + 6x - 16 = 0$

$\Rightarrow (x + 8)(x - 2) = 0$

$x = -8$ (rejected) and $x = 2$

$\therefore y = 2\sqrt{3}$

and point is $(2, 2\sqrt{3})$

Required area = Area of circle – Interior area (or un-shaded area)

Area of circle = $\pi r^2 = \pi(16) = 16\pi$ sq. units

$= 2\int_0^2 \sqrt{6}\sqrt{x}\, dx + 2\int_2^4 \sqrt{16 - x^2}\, dx$

$= 2\sqrt{6} . \frac{2}{3}\left(x^{\frac{3}{2}}\right)_0^2 + 2\left[\frac{x}{2}\sqrt{16 - x^2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$

$= \frac{4\sqrt{6}}{3}\left[2\sqrt{2}\right] + 2\left[\left(0 + 8\sin^{-1}(1)\right) - \left(2\sqrt{3} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right]$

$= \frac{8\sqrt{12}}{3} + 2\left[8 . \frac{\pi}{2} - 2\sqrt{3} - 8 . \frac{\pi}{6}\right]$

$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8\pi}{3}$

$= \left(\frac{16\sqrt{3}}{3} - 4\sqrt{3}\right) + \left(8\pi - \frac{8\pi}{3}\right)$

$= \frac{4\sqrt{3}}{3} + \frac{16\pi}{3}$ sq. units

Now required area = $16\pi - \left[\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right]$

$= 16\pi - \frac{16\pi}{3} - \frac{4\sqrt{3}}{3}$

$= \frac{32\pi}{3} - \frac{4\sqrt{3}}{3}$ sq. units ans.

Q.4) Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Sol.4) 1) $4x^2 + 4y^2 = 9$ or $x^2 + y^2 = \frac{9}{4}$

(.) circle

(.) center $(0,0)$

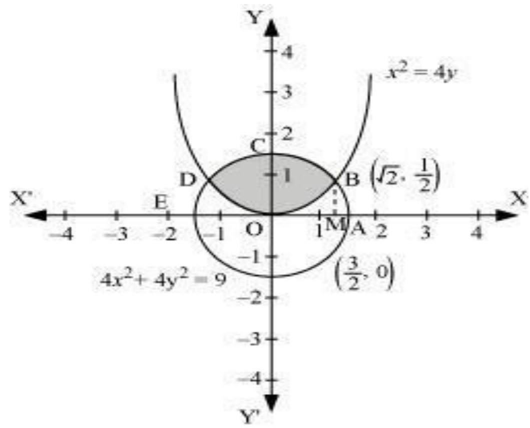
(.) radius = $\frac{3}{2}$

2) $x^2 = 4y$

(.) parabola

(.) vertex $(0,0)$

(.) radius = $\frac{3}{2}$



Intersecting point:

Put $x^2 = 4y$ in $4x^2 + 4y^2 = 9$

$\Rightarrow 16y + 4y^2 = 9$

$\Rightarrow 4y^2 + 16y - 9 = 0$

$\Rightarrow 4y^2 + 18y - 2y - 9 = 0$

$\Rightarrow (2y + 9)(2y - 1) = 0$

$\Rightarrow y = \frac{-9}{2}$ (rejected) and $y = \frac{1}{2}$

$\therefore x = \sqrt{2}$

Point $\left(\sqrt{2}, \frac{1}{2}\right)$

Required area = $2\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\, dx$

$= 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{x^3}{12}\right]_0^{\sqrt{2}}$

$= 2\left[\left(\frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12}\right) - (0)\right]$

$= \frac{\sqrt{2}}{2} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{3}$

$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ sq. unit ans.

Q.5) Find the area lying above x-axis and included the circle $4x^2 + 4y^2 = 32x$ and $y^2 = 4x$

Sol.5) 1) $4x^2 + 4y^2 = 32x$

(.) circle

(.) center $(4,0)$

(.) radius = 4

$\Rightarrow x^2 + y^2 = 8x$

$\Rightarrow x^2 - 8x + y^2 = 0$

$\Rightarrow (x - 4)^2 - 16 + y^2 = 0$

$\Rightarrow (x - 4)^2 + y^2 = 16$

2) $y^2 = 4x$

(.) parabola

(.) vertex (0,0)
(.) open $+ve$ x-axis

Intersection point

Put $y^2 = 4x$ in $x^2 + y^2 = 8x$

$\Rightarrow x^2 + 4x = 8x$

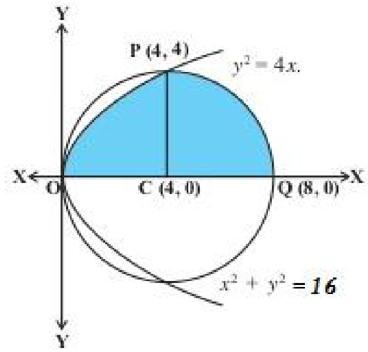
$\Rightarrow x^2 - 4x = 0$

$\Rightarrow x(x-4) = 0$

$x = 0 \; ; \; x = 4$

$\therefore$ points (0,0) & (4,4)

$y = 0 \; ; \; y = 4$



Required area = $\int_0^4 2\sqrt{x}\,dx + \int_4^8 \sqrt{16-(x-4)^2}\,dx$

$= 2 . \frac{2}{3}\left(x^{\frac{3}{2}}\right)_0^4 + \left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2} + 8\sin^{-1}\left(\frac{x-}{4}\right)\right]_4^8$

$= \frac{4}{8}(8-0) + \left[\left(0 + 8\sin^{-1}(1)\right) - (0+0)\right]$

$= \frac{32}{3} + 8 . \frac{\pi}{2}$

$= \frac{32}{3} + 4\pi$ sq. units ans.

Q.6) Find the area of the region in 1st quadrant enclosed by x-axis, line $x = \sqrt{3}y$ and curve $y = \sqrt{4-x^2}$.

Sol.6) 1) $x = \sqrt{3}y$

(.) line passing through (0,0)
(.) 30 with x-axis

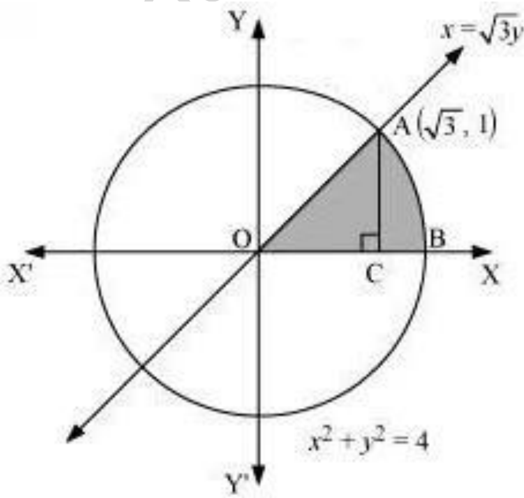
2) $y = \sqrt{4-x^2}$

$\Rightarrow y^2 = 4 - x^2$

$\Rightarrow x^2 + y^2 = 4$

(.) circle
(.) center (0,0)
(.) radius = 2

3) x-axis



Intersection point:

Put $x = \sqrt{3}y$ in $x^2 + y^2 = 4$

$\Rightarrow 3y^2 + y^2 = 4$

$\Rightarrow 4y^2 = 4$

$\Rightarrow y = 1$

$\therefore x = \sqrt{3}$ and point is $(\sqrt{3}, 1)$

Required area = $\int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^2 \sqrt{4-x^2}\,dx$

$= \left(\frac{x^2}{2\sqrt{3}}\right)_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^2$

$= \left(\frac{3}{2\sqrt{3}} - 0\right) + \left[\left(0 + 2\sin^{-1}(1)\right) - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{3}{2\sqrt{3}} + 2 . \frac{\sqrt{3}}{2} - 2 . \frac{\pi}{3}$

$= \frac{3\sqrt{3}}{6} - \frac{\sqrt{3}}{2} + \pi - \frac{2\pi}{3}$

$= \frac{3\sqrt{3}-3\sqrt{3}}{6} + \frac{\pi}{3}$

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$= \left(\frac{x^2}{2\sqrt{3}}\right)_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + 2sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^2$

Required area = $\frac{\pi}{3}$ sq. units ans.

Q.7) Find the area of the region $\{(x,y): |x-1| \le y \le \sqrt{5-x^2}\}$.

Sol.7) 1) $y \le \sqrt{5-x^2}$

$\Rightarrow y^2 \le 5 - x^2$

(.) circle

(.) center $(0,0)$

(.) radius $\sqrt{5}$

(.) solution: Inside the circle

2) $y \ge |x-1|$

(.) $y \ge x - 1$; $x - 1 \ge 0 \Rightarrow x \ge 1$

line

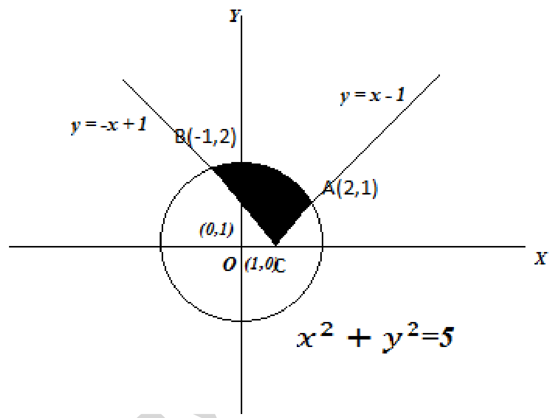
points $(0,-1)$ & $(1,0)$

solution: towards the origin

(.) $y \ge -x+1$; $x-1 < 0 \Rightarrow x < 1$

points $(0,1)$ & $(1,0)$

solution: away from the origin



Intersection point:

Put $y = x - 1$ in $x^2 + y^2 = 5$

$\Rightarrow x^2 + (x-1)^2 = 5$

$\Rightarrow x^2 + x^2 - 2x + 1 = 5$

$\Rightarrow 2x^2 - 2x - 4 = 0$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

$x = 2$ and $x = -1$

$y = 1$ and $y = 2$

$\therefore$ points are (2, 1) and (−1, 2)

Required Area:

$= \int_{-1}^{1} \sqrt{5-x^2} - (-x+1)\ dx + \int_{1}^{2} \sqrt{5-x^2} - (x-1)\ dx$

$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{x^2}{2} - x\right]_{-1}^{1} + \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) - \frac{x^2}{2} + x\right]_{1}^{2}$

$= \left[\left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{2} - 1\right)\right] - \left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{2} + 1\right) +$

$\left[\left(1 + \frac{5}{2}.\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 2 + 2\right)\right] - \left[\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{2} + 1\right]$

$= \left[1 + \frac{5}{2}.\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{2} + 1 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{3}{2}\right] + \left[1 + \frac{5}{2}.sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 1 - \frac{5}{2}.\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right]$

$= 2 - 2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{1}{2}$

Required area = $\frac{1}{2}\left[5\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + 5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 1\right]$ sq. units ans.

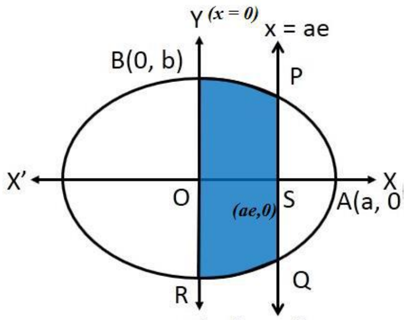
Q.8) Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = 0$ and line $x = ae$ where $b^2 = a^2(1-e)$.

Sol.8) 1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

2) $x = 0$; y-axis

$x = ae$; a line parallel to y-axis at $(ae, 0)$

3)



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Required area = $2\int_0^{ae}\frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= 2\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^1\left(\frac{x}{a}\right)\right]_0^{ae}$

$= \frac{2b}{a}\left[\left(\frac{ae}{2}.\sqrt{a^2-a^2e^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{ae}{a}\right)\right)-(0)\right]$

$= \frac{2b}{b}\left[\frac{ae}{2}.a\sqrt{1-e^2}+\frac{a^2}{2}.\sin^{-1}(e)\right]$

$= \frac{2b}{a}.\frac{a^2}{2}\left[e\sqrt{1-e^2}+\sin^{-1}(e)\right]$

Required area = $ab\left[e\sqrt{1-e^2}+\sin^{-1}e\right]$ sq. units ans.

Q.9) Find the area of the smaller region bounded by the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1.$

Sol.9) 1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(.) ellipse

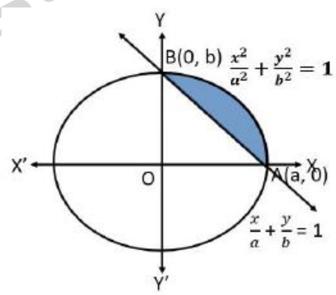
(.) horizontal (let $a > b$)

(.) vertices $(a,0)$ & $(0,b)$

2) $\frac{x}{a}+\frac{y}{b}=1$

(.) line

(.) points $(a,0)$ & $(0,b)$



Required area

$= \int_0^a\frac{b}{a}\sqrt{a^2-x^2}-\left(\frac{b}{a}(a-x)\right)dx$ $\left\{\text{for ellipse} = \frac{\text{b}}{\text{a}}\left(\sqrt{\text{a}^2-\text{x}^2}\right)\right\}$

$= \frac{b}{a}\int_0^a\sqrt{a^2-x^2}-a+x\,dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}-ax+\frac{x^2}{2}\right]_0^a$

$= \frac{b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}(1)-a^2+\frac{a^2}{2}\right)-(0+0-0+0)\right]$

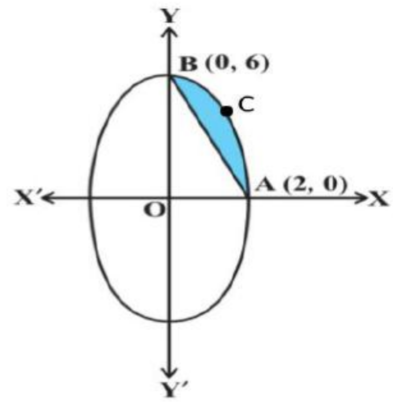
$= \frac{b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2}-\frac{a^2}{2}\right]$

$= \frac{b}{a}\left(\frac{a^2\pi}{4}-\frac{a^2}{2}\right)$

$= \frac{a^2b}{2a}\left(\frac{\pi}{2}-1\right)$

Required area = $\frac{a^2b}{2}\left(\frac{\pi}{2}-1\right)$ sq. units ans.

Q.10) AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that $OA = 2$ and $OB = 6$. Find the area between the arc AB and chord AB.

Sol.10) 1)	$9x^2 + y^2 = 36$ (Divide by 36) $\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ here $a = 2$ & $b = 3$ and $b > a$ (.) ellipse (.) verticals 
2)	$A(2,0)$ & $B(0,6)$ Equation of chord A $y - 0 = \frac{6-0}{0-2}(x-2)$ $\Rightarrow y = -3(x-2)$ $\Rightarrow y = -3x + 6$ Required area = $\int_0^2 \frac{6}{2}\sqrt{4-x^2} - (-3x+6)\, dx$ $= 3\int_0^2 \sqrt{4-x^2} + x - 2\, dx$ $= 3\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x^2}{2} - 2x\right]_0^2$ $= 3[(0 + 2\sin^{-1}(1) + 2 - 4) - (0)]$ $= 3\left[2 . \frac{\pi}{2} - 2\right]$ Required area = $(3\pi - 6)$ sq. units ans.