APPLICATION OF INTEGRALS
Class 12th

Q.1) Find the area bounded by the curves $y = 2x - x^2$ and line $y = -x$

Sol.1) 1) $y = 2x - x^2$ (shifting parabola)

$\Rightarrow x^2 - 2x = -y$

$\Rightarrow (x - 1)^2 - 1 = -y$

$\Rightarrow (x - 1)^2 = -y + 1$

$\Rightarrow (x - 1)^2 = -(y - 1)$

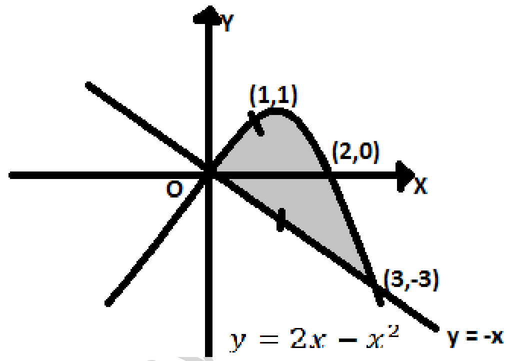
(.) parabola (shifting)

(.) vertex $(1, -1)$

(.) open towards $-ve$ y-axis

2) $y = -x$

(.) line passes through $(0,0)$



Intersections point:

Solving $y = 2x - x^2$ and $y = -x$

$\Rightarrow -x = 2x - x^2$

$\Rightarrow x^2 - 3x = 0$

$x(x - 3) = 0$

$\Rightarrow x = 0 \; and \; x = 3$

$y = 0 \; and \; x = -3$

$\therefore$ points $(0,0)$and $(3, -3)$

Required area = $\int_0^3 [2x - x^2 - (-x)]\, dx$

$= \int_0^3 (3x - x^2)\, dx$

$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3$

$= \left(\frac{27}{2} - \frac{27}{3}\right) - (0)$

$= \frac{81-54}{6}$

$= \frac{27}{6} = \frac{9}{2}$

$\therefore$ Required area = $\frac{9}{2}$ sq. units ans.

Q.2) Find the area bounded by curves $y = 6x - x^2$ and $y = x^2 - 2x$.

Sol.2) 1) $y = 6x - x^2$

$\Rightarrow x^2 - 6x = -y$

$\Rightarrow (x - 3)^2 - 9 = -y$

$\Rightarrow (x - 3)^2 = -y + 9$

$\Rightarrow (x - 3)^2 = -(y - 9)$

(.) vertex (3,9)

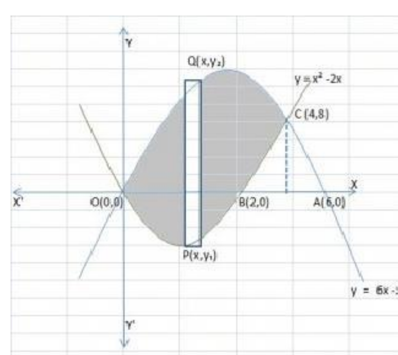
(.) shifting parabola

(.) open towards $-ve$ y-axis

(Imp.) Intersection point of this parabola with x-axis $(y = 0)$.

Put $y = 0$ in $y = 6x - x^2$

$\Rightarrow x^2 - 6x = 0$



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2)

$\Rightarrow\ x(x - 6) = 0$

$x = 0\ and\ x = 6$

$y = x^2 - 2x$

$\Rightarrow x^2 - 2x = y$

$\Rightarrow (x - 1)^2 - 1 = y$

$\Rightarrow (x - 1)^2 = (y + 1)$

(.) shifting parabola

(.) vertex $(1, -1)$

(.) open towards $+ve\ y$-axis

Intersection point of this parabola with $x - axis\ (y = 0)$

Put $y = 0$ in $y = x^2 - 2x$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x - 2) = 0$

$x = 0$ and $x = 2$

Required area = $\int_0^4 [(6x - x^2) - (x^2 - 2x)]\, dx$

$= \int_0^4 (8x - 2x^2)\, dx$

$= \left[4x^2 - \frac{2x^3}{3}\right]_0^4$

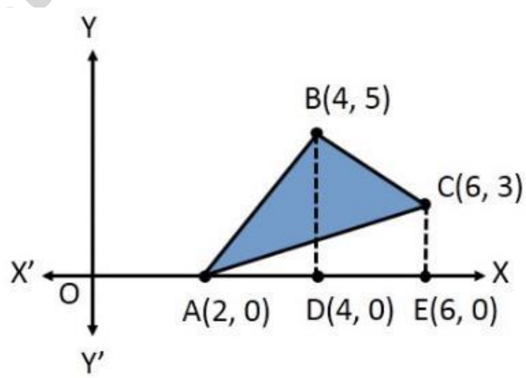
$= \left[64 - \frac{128}{3}\right] - [0]$

Required = $\frac{64}{3}$ sq. units. ans.

Q.3) Find the area bounded by the triangle whose vertices are $A(2,0), B(4,5), \& C(6,3)$.

Sol.3) Vertices are $A(2,0)\ \&\ B(4,5), C(6,3)$

Equation of side AB (two point form)



$y - 0 = \frac{5-0}{4-2}(x - 2)$

$y = \frac{5x-10}{2}$

equation of side BC:

$y - 5 = \frac{3-5}{6-4}(x - 4)$

$y - 5 = -1(x - 4)$

$\Rightarrow\ y - 5 = -x + 4$

$\Rightarrow\ y = -x + 9$

Equation of side AC

$y - 0 = \frac{3-0}{6-2}(x - 2)$

$y = \frac{3}{4}(x - 2)$

Required area = $\int_2^4 \left(\frac{5x-10}{2}\right) - \left(\frac{3x-6}{4}\right) dx + \int_4^6 (-x + 9) - \left(\frac{3x-6}{4}\right) dx$

$= \frac{1}{4}\int_2^4 (10x - 20 - 3x + 6)\, dx + \frac{1}{4}\int_4^6 (-4x + 36 - 3x + 6)\, dx$

$= \frac{1}{4}\int_2^4 (7x - 14)\, dx + \frac{1}{4}\int_4^6 (-7x + 42)\, dx$

$= \frac{7}{4}\int_2^4 (x - 2)\, dx + \frac{7}{4}\int_4^6 -x + 6\, dx$

$= \frac{7}{4}\left[\frac{x^2}{2} - 2x\right]_2^4 + \frac{7}{4}\left[\frac{-x^2}{2} + 6x\right]_4^6$

$= \frac{7}{4}[(8 - 8) - (2 - 4)] + \frac{7}{4}[(-18 + 36) - 8 + 24)$

$= \frac{7}{4}(2) + \frac{7}{4}(2)$

$= \frac{7}{2} + \frac{7}{2}$

$= 7$

$\therefore$ Required area $= 7\ sq.units$ ans.

Q.4) Find the area bounded by the lines $2x + y = 4, 3x - 2y = 6$ and $x - 3y + 5 = 0$.

Sol.4) Given,

$2x + y = 4$ ......(1)

$3x - 2y = 6$ ......(2)

$x - 3y = -5$ ....(3)

Solving (1) & (2)

$6x + 3y = 12$

$6x - 4y = 12$

$7y = 0$

$y = 0$

$\therefore x = 2$

$A(2, 0)$

Solving (2) & (3)

$3x - 2y = 6$

$3x - 9y = -15$

$7y = 21$

$y = 3$

$\therefore x = 4$

$B(4, 3)$

Solving (1) & (3)

$2x + y = 4$

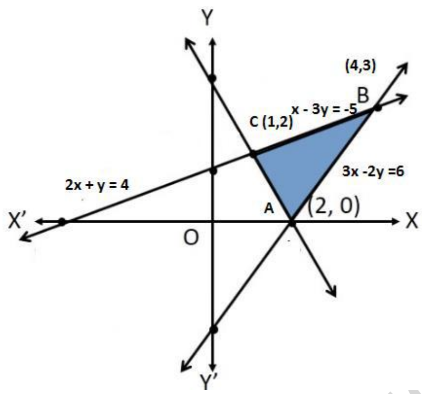
$2x - 6y = -10$

$7y = 14$

$y = 2$

$\therefore x = 1$

$C(1, 2)$



Now equation of AB = eq. (2) i.e. $3x - 2y = 6$

Equation of BC = eq. (3) i.e. $x - 3y = -5$

& equation of AC = eq. (1) i.e. $2x + y = 4$

Required area = $\int_1^2 \left(\frac{x+5}{3}\right) - (4 - 2x)\, dx + \int_2^4 \left(\frac{x+5}{3}\right) - \left(\frac{3x-6}{2}\right) dx$

$= \frac{1}{3}\int_1^2 (x + 5 - 12 + 6x)\, dx + \frac{1}{6}\int_2^4 (2x + 10 - 9x + 18)\, dx$

$= \frac{1}{3}\int_1^2 (7x - 7)\, dx + \frac{1}{6}\int_2^4 (28 - 7x)\, dx$

$= \frac{7}{3}\int_1^2 (x - 1)\, dx + \frac{7}{6}\int_2^4 (4 - x)\, dx$

$= \frac{7}{3}\left[\frac{x^2}{2} - x\right]_1^2 + \frac{7}{6}\left[4x - \frac{x^2}{2}\right]_2^4$

$= \frac{7}{3}\left[(2 - 2) - \left(\frac{1}{2} - 1\right)\right] + \frac{7}{6}[(16 - 8) - (8 - 2)]$

$= \frac{7}{3}\left(\frac{1}{2}\right) + \frac{7}{6}(2)$

$= \frac{7}{6} + \frac{14}{6} = \frac{21}{6} = \frac{7}{2}$

$\therefore$ Required area = $\frac{7}{2}$ sq. units ans.

Q.5) Find the area of region bounded by the line $y = 3x + 2$, x-axis, $x = -1$ and $x = 1$.

Sol.5) 1) $y = 3x + 2$

(.) line passing through points (0,2) and $\left(\frac{-2}{3}, 0\right)$

2) $x -$ axis

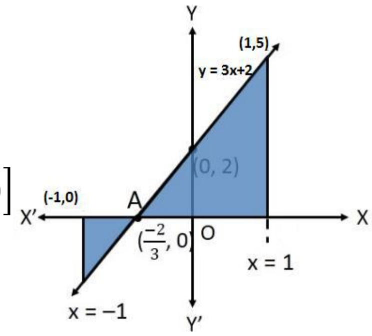
$x = -1$

3) (.) line parallel to y-axis at $(-1, 0)$

$x = 1$

4) (.) line parallel to y-axis at $(1,0)$

Required area = $\int_{-1}^{\frac{-2}{3}} 0 - (3x+2)\,dx + \int_{\frac{-2}{3}}^{1} (3x+2) - 0\,dx$



$$= -\left[\frac{3x^2}{2} + 2x\right]_{-1}^{\frac{-2}{3}} + \left[\frac{3x^2}{2} + 2x\right]_{\frac{-2}{3}}^{1}$$

$$= \left[\left(\frac{3}{2}.\frac{4}{9} - \frac{4}{3}\right) - \left(\frac{3}{2} - 2\right)\right] + \left[\left(\frac{3}{2} + 2\right) - \left(\frac{2}{3} - \frac{4}{3}\right)\right]$$

$$= \left[\frac{-2}{3} + \frac{1}{2}\right] + \left[\frac{7}{2} + \frac{2}{3}\right]$$

$$= -\left[\frac{-4+3}{6}\right] + \left[\frac{21+4}{6}\right]$$

$$= \frac{1}{6} + \frac{25}{6} = \frac{26}{6} = \frac{13}{3}$$

$\therefore$ Required area = $\frac{13}{3}$ sq. units ans.

Q.6) Sketch the graph $y = |x + 3|$.

Evaluate $\int_{-6}^{0} |x+3|\,dx$. What does this value represent on the graph?

Sol.6) We have, $y = |x+3|$

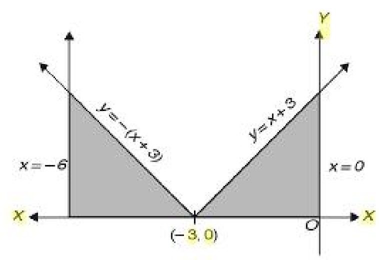
$$y = \begin{cases} x+3 & x+3 \geq -3 \quad x \geq -3 \\ -(x+3) & x+3 < 0 \quad x < -3 \end{cases}$$

(.) $y = x + 3$; $x \geq -3$

Point (0,3) and $(-3,0)$

(.) $y = -x - 3$; $x < -3$

Point $(0, -3)$ and $(-3,0)$



Now $\int_{-6}^{0} |x+3|\,dx = -\int_{-6}^{-3} (x+3)\,dx + \int_{-3}^{0} (x+3)\,dx$

$$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= -\left[\left(\frac{9}{2} - 9\right) - \left(\frac{36}{2} - 18\right)\right] + \left[(0) - \left(\frac{9}{2} - 9\right)\right]$$

$$= -\left[-\frac{9}{2} - 0\right] + \left[\frac{9}{2}\right]$$

$$= \frac{9}{2} + \frac{9}{2} = 9$$

$\therefore \int_{-6}^{0} |x+3|\,dx = 9$ ans.

This value represent the area of the shaded region in t he graph.

Q.7) Find the area bounded by the curve $|x| + |y| = 1$.

Sol.7) We have $|x| + |y| = 1$

This curve has four lines

1) $x + y = 1 : x \geq 0$ and $y \geq 0$

Point (0,1) and (1,0)

2) $-x + y = 1$ $x < 0$ and $y \geq 0$

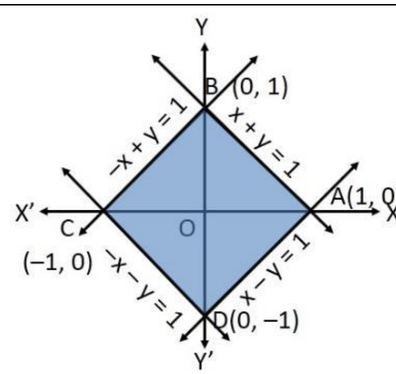
Points (0,1) and $(-1,0)$

3) $x - y = 1$; $x \geq 0$ and $y < 0$

Point $(0, -1)$ & (1,0)

4) $-x - y = 1$; $x < 0$ and $y < 0$

Point $(0, -1)$ & $(-1,0)$



Required area=$\int_{-1}^{0}(1+x)-0\,dx+\int_{0}^{1}(1-x)-0\,dx+\int_{0}^{1}0-(x-1)\,dx+\int_{-1}^{0}0-(-x-1)\,dx$

$=\left[x+\frac{x^2}{2}\right]_{-1}^{0}+\left[x-\frac{x^2}{2}\right]_{0}^{1}+\left[\frac{-x^2}{2}+x\right]_{0}^{1}+\left[\frac{x^2}{x}+x\right]_{-1}^{0}$

$=\left[(0)-\left(-1+\frac{1}{2}\right)\right]+\left[\left(1-\frac{1}{2}\right)-(0)\right]+\left[\left(\frac{-1}{2}+1\right)-0\right]+\left[(0)-\left(\frac{1}{2}-1\right)\right]$

$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$

$=2$

$\therefore$ Req. Area = 2 sq. units

Q.8) Find the area bounded by the curves $y=2+|x+2|$; $x=-4$ and $x=4$ and $y=0$.

Sol.8)

1) $y = 2+|x+2|$

(.) $y = 2+(x+2)$; $x+2\geq 0$

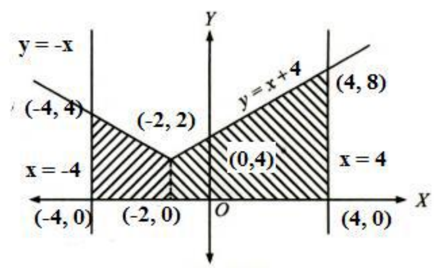
$\Rightarrow x\geq -2$; $y=x+4$

Points $(0,4)$ and $(-4,0)$

(.) $y=2-(x+2)$; $x+2<0$

$\Rightarrow x<-2$; $y=-x$

line passing through $(0,0)$ and 45° with x-axis



2) $x=-4$

(.) line parallel to y-axis at $(-4,0)$

3) $x = 4$

(.) line parallel to y-axis at $(4,0)$

4) $y=0$

(.) equation of x-axis

Required area = $\int_{-4}^{-2}(-x)\,dx+\int_{-2}^{4}(x+4)\,dx$

$=-\left(\frac{x^2}{2}\right)_{-4}^{-2}+\left(\frac{x^2}{2}+4x\right)_{-2}^{4}$

$=-[2-8]+[(8+16)-(2-8)]$

$=6+24+6$

$=32$

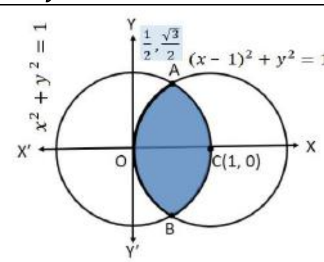
$\therefore$ Required area = $32\ sq.units$ ans.

Q.9) Find the area bounded by the curves $x^2+y^2=1$ and $(x-1)^2+y^2=1$.

Sol.9)

1) $x^2+y^2=1$

(.) circle : center $(0,0)$ and radius = 1



2) $(x-1)^2+y^2=1$

(.) circle : center $(1,0)$ and radius = 1

Required Area

$=2\int_{0}^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+2\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx$

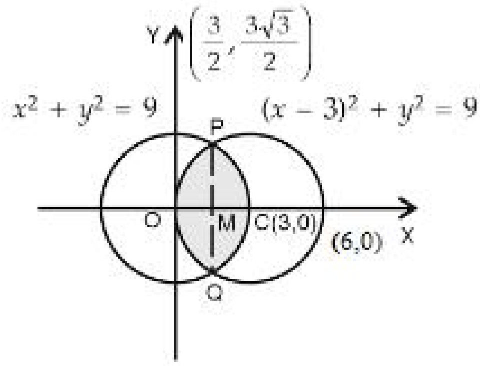
$=2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}sin^{-1}(x-1)\right]_{0}^{\frac{1}{2}}+2\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_{\frac{1}{2}}^{1}$

$=2\left[\left(-\frac{1}{4}.\frac{\sqrt{3}}{2}+\frac{1}{2}\sin^{-1}\left(\frac{-1}{2}\right)\right)\right]-\left[0+\frac{1}{2}\sin^{-1}(-1)\right]+2\left[\left(0+\frac{1}{2}\sin^{-1}(1)\right)-\left(\frac{1}{4}.\frac{\sqrt{3}}{2}+\frac{1}{2}\sin^{-1}\frac{1}{2}\right)\right]$

$=2\left[\left(-\frac{\sqrt{3}}{8}-\frac{1}{2}.\frac{\pi}{6}\right)-\left(\frac{-1}{2}.\frac{\pi}{2}\right)\right]+2\left[\left(\frac{1}{2}.\frac{\pi}{2}\right)-\left(\frac{\sqrt{3}}{8}+\frac{1}{2}.\frac{\pi}{6}\right)\right]$

$=2\left[-\frac{\sqrt{3}}{8}-\frac{\pi}{12}+\frac{\pi}{4}\right]+2\left[\frac{\pi}{4}-\frac{\sqrt{3}}{8}-\frac{\pi}{12}\right]$

$=-\frac{\sqrt{3}}{4}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}$

	$= \pi - \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ $\therefore$ Required area = $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq. units ans.
Q.10)	Find the area bounded by the curves $(x-3)^2 + y^2 \geq 9$ and $x^2 + y^2 \leq 9$.
Sol.10) 1) 2)	$(x-3)^2 + y^2 \geq 9$ (.) circle : center = $(3,0)$ and radius = 3 (.) solution : outside the circle $x^2 + y^2 \leq 9$ (.) circle : center $(0,0)$ and radius = 3 (.) solution : Inside the circle  **Intersection point** Solving $x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$ Put $y^2 = 9 - x^2$ in $(x-3)^2 + y^2 = 9$ $\Rightarrow (x-3)^2 + 9 - x^2 = 9$ $\Rightarrow x^2 + 9 - 6x + 9 - x^2 = 9$ $\Rightarrow 6x = 9$ $x = \frac{3}{2}$ $\therefore y = \frac{3\sqrt{3}}{2}$ points $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ Now required = Area of circle – [Area of region $(A + B)$] Area of circle = $\pi r^2 = \pi(3)^2 = 9\pi$ sq. units Area of region $(A + B)$ = $2\int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\, dx + 2\int_{\frac{3}{2}}^{3} \sqrt{9-x^2}\, dx$ $= 2\left[\frac{(x-3)}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^{\frac{3}{2}} + 2\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{2}}^{3}$ $= 2\left[\left(\frac{-3}{4}\cdot\frac{3\sqrt{3}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{-1}{2}\right)\right) - \left(0 + \frac{9}{2}\sin^{-1}(-1)\right)\right] + 2\left[\left(0 + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{3}{4}\cdot\frac{3\sqrt{3}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)\right]$ $= 2\left[\left(\frac{-9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6}\right) - \left(\frac{-9}{2}\cdot\frac{\pi}{2}\right)\right] + 2\left[\frac{9}{2}\cdot\frac{\pi}{2} - \frac{9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6}\right]$ $= \frac{-9\sqrt{3}}{4} - \frac{9\pi}{6} + \frac{9\pi}{2} + \frac{9\pi}{2} - \frac{9\sqrt{3}}{4} - \frac{9\pi}{6}$ $= 9\pi - \frac{9\pi}{3} - \frac{9\sqrt{3}}{2}$ $= \frac{18\pi}{3} - \frac{9\sqrt{3}}{2}$ sq. units Now required area = $9\pi - \left(\frac{18\pi}{3} - \frac{9\sqrt{3}}{2}\right)$ $= \left(\frac{9\pi}{3} + \frac{9\sqrt{3}}{2}\right)$ sq. units ans.