UNIT V

MOTION OF SYSTEMS OF PARTICLES AND RIGID BODY

- **Centre of mass** of a body is a point where the entire mass of the body can be supposed to be concentrated.

- For a system of *n*-particles, the centre of mass is given by

$$\vec{r} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \ldots\ldots + m_n\vec{r_n}}{m_1 + m_2 + m_3 + \ldots\ldots + m_n} = \frac{\sum_{i=1}^{i=n} m_i\vec{r_1}}{M}$$

where $M = m_1 + m_2 + \ldots m_n$

- **Torque** $\left(\vec{\tau}\right)$ *The turning effect of a force with respect to some axis, is called moment of force or torque due to the force.* Torque is measured as the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the axis of rotation.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

SI unit of torque is Nm.

- **Angular momentum** $\left(\vec{L}\right)$. *It is the rotational analogue of linear momentum and is measured as the product of the linear momentum and the perpendicular distance of its line action from the axis of rotation.*

If $\vec{P}$ is linear momentum of the particle and $\vec{r}$ its position vector, then angular momentum of the particle, $\vec{L} = \vec{r} \times \vec{p}$

SI unit of angular momentum is kg m^2 s^{-1}.

- **Relation** between torque and angular momentum :

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

- **Moment of inertia (I).** *The moment of inertia of a rigid body about a given axis is the sum of the products of masses of the various particles* **with** *squares of their respective perpendicular distances from the axis of rotation.*

  $$I = m_1r_1^2 + m_2r_2^2 + m_3r_2^2 + ... + m_nr_n^2 = \sum_{i=1}^{i=n} m_ir_i^2$$

  SI unit of moment of inertia is kg m^2.

- **Radius of gyration (K).** It is defined as the distance of a point from the axis of rotation at which, if whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass with respect to the same axis

  $$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + ... + r_n^2}{n}} \quad \text{and } I = MK^2.$$

  SI unit of radius of gyration is m.

- **Theorem of perpendicular axes.** It states that the moment of inertia of a 2-d object about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point, where the perpendicular axis passes through the **plane.**

  $$I_z = I_x + I_y$$

  where X and Y-axes lie in the plane of the **object** and Z-axis is perpendicular to its plane and passes through the point of intersection of X and Y axes.

- **Theorem of parallel axes.** It states that the moment of inertia of a rigid body about any axis is equal to moment of inertia of the body about a parallel axis through its centre of mass plus the product of mass of the body and the square of the perpendicular distance between the axes.

  $I = I_c + M\,h^2$, where I_c is moment of inertia of the body about an axis through its centre of mass and h is the perpendicular distance between the two axes.

- **Moment of inertia of some object :-**

S. No.	Body		Axis of rotation	Moment of Inertia (I)
•	Uniform circular ring of mass M and radius R	(i)	about an axis passing through centre and perp. to its plane.	MR^2
		(ii)	about a diameter.	$\frac{1}{2}MR^2$
		(iii)	about a tangent in its own plane.	$\frac{3}{2}MR^2$
		(iv)	about a tangent $\perp$ to its plane.	$2\ MR^2$
•	Uniform circular disc of mass M and radius R.	(i)	about an axis passing through centre and perp. to its plane.	$\frac{1}{2}MR^2$
		(ii)	about a diameter.	$\frac{1}{4}MR^2$
		(iii)	about a tangent in its own plane.	$\frac{5}{4}MR^2$
		(iv)	about a tangent $\perp$ to its plane.	$\frac{3}{2}MR^2$
•	Solid sphere of radius R and mass M	(i)	about its diameter.	$\frac{2}{5}MR^2$
		(ii)	about a tangential axis.	$\frac{7}{5}MR^2$
•	Spherical shell of radius R and mass M.	(i)	about is diameter.	$\frac{2}{3}MR^2$
		(ii)	about a tangential axis.	$\frac{5}{3}MR^2$
•	Long thin rod of	(i)	about an axis through	$\frac{ML^2}{12}$

length L.		C.G. and $\perp$ to rod.	
	(ii)	about an axis through one end and $\perp$ to rod.	$\frac{ML^2}{3}$

- **Law of conservation of angular momentum.** If no external torque acts on a system, the total angular momentum of the system remains unchanged.

  $I\vec{\omega} =$ constant vector or $I_1\omega_1 = I_2\omega_2$, provided no external torque acts on the system.

- For **translational equilibrium** of a rigid body, $\vec{F} = \sum_i \vec{F}_{ext} = 0$

- For **rotational equilibrium** of a rigid body, $\vec{\tau} = \sum_i \vec{\tau}_{ext} = 0$

- **Analogy between various quantities describing linear motion and rotational motion.**

S.No.	linear motion	S.No.	Rotation motion
•	Distance/displacement (s)	1.	Angle or angular displacement (θ)
•	Linear velocity, $v = \frac{dx}{dt}$	2.	Angular velocity, $\omega = \frac{d\theta}{dt}$
•	Linear acceleration, $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$	3.	Angular acceleration, $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
•	Mass (m)	4.	Moment of interia (I)
•	Linear momentum, $p = m\,v$	5.	Angular momentum, $L = I\omega$
•	Force, $F = m\,a$	6.	Torque, $\tau = I\alpha$
•	Also, force $F = \frac{dp}{dt}$	7.	Also, torque, $\tau = \frac{dL}{dt}$
•	Translational KE, $K_T = \frac{1}{2}mv^2$	8.	Rotational K.E., $K_R = \frac{1}{2}I\omega^2$
•	Work done, $W = F\,s$	9.	Work done, $W = \tau\,\theta$
•	Power, $P = F\,v$	10.	Power, $P = \tau\,\omega$
•	(Principle of conservation of linear momentum)		(Principle of conservation of angular momentum)

• Equations of translatory motion	11.	Equations of rotational motion
(i) v = u + at (ii) $s = ut + \frac{1}{2}at^2$		$\omega_2 = \omega_1 + \alpha t$
(iii) $v^2 - u^2 = 2as$,		(ii) $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
		(iii) $\omega_2^2 - \omega_1^2 = 2\alpha\theta$

Motion of a body rolling without slipping on an inclined plane acceleration

$$a = \frac{mg\sin\theta}{m + I/r^2}$$

Kinetic energy of a rolling body is

E_K = K.E of translation (K_T) + K.E. of rotation (K_e)

$$E_K = \frac{1}{2}M\nu^2 + \frac{1}{2}Iw^2$$

1 MARK QUESTIONS

1. What is a rigid body?
2. State the principle of moments of rotational equilibrium.
3. Is centre of mass of a body necessarily lie inside the body? Give any example
4. Can the couple acting on a rigid body produce translatory motion?
5. Which component of linear momentum does not contribute to angular momentum?
6. A system is in stable equilibrium. What can we say about its potential energy?
7. Is radius of gyration a constant quantity?
8. Two solid spheres of the same mass are made of metals of different densities. Which of them has a large moment of inertia about the diameter?
9. The moment of inertia of two rotating bodies A and B are I_A and I_B ($I_A > I_B$) and their angular momenta are equal. Which one has a greater kinetic energy?

10. A particle moves on a circular path with decreasing speed. What happens to its angular momentum?

11. What is the value of instantaneous speed of the point of contact during pure rolling?

12. Which physical quantity is conserved when a planet revolves around the sun?

13. What is the value of torque on the planet due to the gravitational force of sun?

14. If no external torque acts on a body, will its angular velocity be constant?

15. Why there are two propellers in a helicopter?

16 A child sits stationary at one end of a long trolley moving uniformly with speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, then what is the effect of the speed of the centre of mass of the (trolley + child) system?

ANSWERS

3. No. example ring

4. No. It can produce only rototory motion.

5. Radial Component

6. P.E. is minimum.

7. No, it changes with the position of axis of rotation.

8. Sphere of small density will have large moment of inertia.

9. $K = \frac{L^2}{2I} \quad \Rightarrow \quad K_B > K_A$

10. as $\vec{L} = \vec{r} \times m\vec{v}$ i : e magnitude $\vec{L}$ decreases but direction remains constant.

11. zero

12. Angular momentum of planet.

13. zero.

14. No. $\omega \alpha \frac{1}{I}$.

15. due to conservation of angular momentum

16. No change in speed of system as no external force is working.

2 MARKS QUESTIONS

1. Show that in the absence of any external force, the velocity of the centre of mass remains constant.

2. State the factors on which the position of centre of mass of a rigid body depends.

3. What is the turning effect of force called for ? On what factors does it depend?

4. State the factors on which the moment of inertia of a body depends.

5. On what factors does radius of gyration of body depend?

6. Why do we prefer to use a wrench of longer arm?

7. Can a body be in equilibrium while in motion? If yes, give an example.

8. There is a stick half of which is wooden and half is of steel. (i) it is pivoted at the wooden end and a force is applied at the steel end at right angle to its length (ii) it is pivoted at the steel end and the same force is applied at the wooden end. In which case is the angular acceleration more and why?

9. If earth contracts to half its radius what would be the length of the day at equator?

10. An internal force can not change the state of motion of centre of mass of a body. How does the internal force of the brakes bring a vehicle to rest?

11. When does a rigid body said to be in equilibrium? State the necessary condition for a body to be in equilibrium.

12. How will you distinguish between a hard boiled egg and a raw egg by spinning it on a table top?

13. What are binary stars? Discuss their motion in respect of their centre of mass.

14. In which condition a body in gravitational field is in stable equilibrium?

15. Give the physical significance of moment of inertia.

ANSWERS

2. (i) Shape of body

   (ii) mass distribution

3. Torque

   Factors (i) Magnitude of force

   (ii) Perpendicular distance of force vector from axis of rotation.

4. (i) Mass of body

   (ii) Size and shape of body

   (iii) Mass distribution w.r.t. axis of rotation

   (iv) position and orientation of rotational axis

5. Mass distribution.

6. to increase torque.

7. Yes, if body has no linear and angular acceleration. Hence a body in uniform straight line motion will be in equilibrium.

8. I (first case) > I(Second case)

   $\because \quad \tau = I\alpha$

   $\Rightarrow \alpha$ (first case) < α (second case)

9. $I_1 = \frac{2}{5}MR^2 \quad \Rightarrow \quad I_2^e = \frac{2}{5}M\left(\frac{R}{2}\right)^2 \quad \Rightarrow \quad I_2 = \frac{I}{4}$

   $L = I_1w_1 = I_2w_2$

or $$I\left(\frac{2\pi}{T_1}\right)=\frac{1}{4}\left(\frac{2\pi}{T_2}\right)$$

or $$T_2=\frac{T_1}{4}=\frac{24}{4}=6 \text{ hours}$$

10. In this case the force which bring the vehicle to rest is friction, and it is an external force.

11. For translation equilibrium

$$\sum \vec{F}_{ext}=0$$

For rotational equilibrium

$$\sum \vec{\tau}_{ext}=0$$

12. For same external torque, angular acceleration of raw egg will be small than that of Hard boiled egg

14. When vertical line through centre of gravity passes through the base of the body.

15. It plays the same role in rotatory motion as the mass does in translatory motion.

3 MARKS QUESTIONS

1. Derive the three equation of rotational motion (i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

under constant angular acceleration. Here symbols have usual meaning.

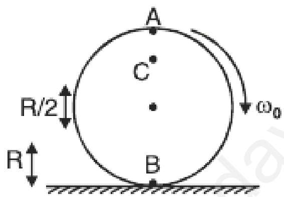
2. Obtain an expression for the work done by a torque. Hence write the expression for power.

3. Prove that the rate of change of angular momentum of a system of particles about a reference point is equal to the net torque acting on the system.

4. Derive a relation between angular momentum, moment of inertia and angular velocity of a rigid body.

5. Show that moment of a couple does not depend on the point about which moment is calculated.

6. A disc rotating about its axis with angular speed ω_0 is placed lightly (without any linear push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the dics shown in figure. Will the disc roll?



7. A uniform circular disc of radius R is rolling on a horizontal surface. Determine the tangential velocity (i) at the upper most point (ii) at the centre of mass and (iii) at the point of contact.

8. Derive an expression for the total work done on a rigid body executing both translational and rotational motions.

9. Prove that the acceleration of a solid cylinder rolling without slipping down an inclined plane is $\frac{2g}{3}\sin\theta$.

10. Show that the angular momentum of a particle is the product of its linear momentum and moment arm. Also show that the angular momentum is produced only by the angular component of linear momentum.

ANSWER

6. For A $V_A = R\omega_0$ in forward direction

For B = $V_B = R\omega_0$ in backward direction

For C $V_C = \frac{R}{2}\omega_0$ in forward direction disc will not roll.

NUMERICALS

1. Three masses 3 kg, 4 kg and 5 kg are located at the corners of an equilateral triangle of side 1m. Locate the centre of mass of the system.

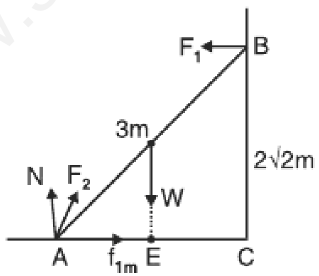
2. Two particles mass 100 g and 300 g at a given time have velocities $10\hat{i} - 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 9\hat{j} + 6\hat{k}$ ms^{-1} respectively. Determine velocity of COM.

3. From a uniform disc of radius R, a circular disc of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of original disc. Locate the centre of gravity of the resultant flat body.

4. The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration (assume the acceleration to be uniform) (ii) How many revolutions does the wheel make during this time?

5. A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm, what is the mass of the matre stick?

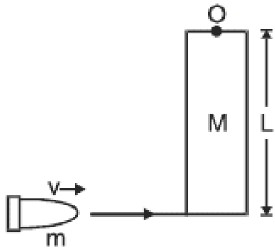
6. A 3m long ladder weighting 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in figure. Find the reaction forces F_1 and F_2 of the wall and the floor.



7. Calculate the ratio of radii of gyration of a circular ring and a disc of the same radius with respect to the axis passing through their centres and perpendicular to their planes.

8. An automobile moves on a road with a speed of 54 kmh^{-1}. The radius of its wheels is 0.35 m. What is the average negative torque transmitted by its brakes to a wheel if the vehicle is brought to rest in 15s? The moment of inertia of the wheel about the axis of rotation is 3 kg m^2.

9. A rod of length L and mass M is hinged at point O. A small bullet of mass m hits the rod, as shown in figure. The bullet get embedded in the rod. Find the angular velocity of the system just after the impact.



10. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10π rad s^{-1}. Which of the two will start to roll earlier? The coefficient of kinetic friction is $\mu_k = 0.2$

ANSWERS

1. $(x, y) = (0.54 \text{ m}, 0.36 \text{ m})$

2. Velocity of COM $= \dfrac{31\hat{i} - 34\hat{j} + 15\hat{k}}{4} \text{ms}^{-1}$

3. COM of resulting portion lies at R/6 from the centre of the original disc in a direction opposite to the centre of the cut out portion.

4. $\alpha = 4\pi$ rad s^{-1}

   $n = 576$

5. $m = 66.0$ g

6. $F_2 = \sqrt{f^2 + N^2}$

   $= \sqrt{34.6^2 + 196^2} = 199.0 \text{ N}$

   If F_2 makes an angle α with the horizontal then

   $\tan\alpha = \dfrac{N}{f} = 5.6568$

   $\alpha = 80°$

7. $\frac{K_{ring}}{K_{disc}} = \frac{R}{R/\sqrt{2}} = \frac{\sqrt{2}}{1}$

8. $\alpha = \frac{w - w_o}{t} = -\frac{1}{0.35} \text{rads}^{-2}$

$\tau = I\alpha = -8.57 \text{ kgm}^2\text{s}^{-2}$

9. Using conservation of angular momentum

$L_{initial} = L_{final}$

$M\ V\ L = I\omega$

or $M\ V\ L = \frac{M+3m}{3} L^2 w$

or $\omega = \frac{3mv}{(M+3m)L}$

10. The disc begins to roll earlier than the ring.

5 MARK QUESTIONS

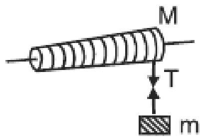
1. Obtain the expression for the linear acceleration of a cylinder rolling down an inclined plane and hence find the condition for the cylinder to roll down without slipping.

2. Prove the result that the velocity V of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by

$$v^2 = \frac{2gh}{1 + \frac{k^2}{R^2}}$$

where K = Radius of gysation of body about its symmetry axis, and R is radius of body. The body starts from rest at the top of the plane.

3. A light string is wound round a cylinder and carries a mass tied to it at the free end. When the mass is released, calculate.



(a) the linear acceleration of the decending mass

(b) angular acceleration of the cylinder

(c) Tension in the string.

4. State the theorem of

(i) perpendicular axis (ii) parallel axis.

Find the moment of inertia of a rod of mass M and length L about and axis perpendicular to it through one end. Given the moment of inertia about an axis perpendicular to rod and through COM is $\frac{1}{12}ML^2$