SEQUENCE AND SERIES

Class XI

Q.1) If f is a function satisfying $f(x+y)=f(x).f(y)$ such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$. Find value of n.

Sol.1) We have, $f(x+y)=f(x)f(y)$

$f(1)=3$

$$\sum_{x=1}^{n} f(x)=120$$

Now $\sum_{x=1}^{n} f(x)=f(1)+f(2)+f(3)+\ldots\ldots\ldots n$ terms$=120$

Now $f(2)=f(1+1)=f(1).f(1)=(3)(3)=9$

$f(3)=f(1+2)=f(1).f(2)=(3)(9)=27$

$f(4)=f(1+3)=f(1).f(3)=(3)(27)=81$

$\therefore$ series becomes

$3+9+27+81+\ldots\ldots\ldots n\ terms=120$

Clearly it is a G.P. with $a=3\ \&\ r=3$

$$3\left[\frac{3^{n-1}-1}{3-1}\right]=120$$

$\Rightarrow 3\frac{(3^n-1)}{2}=120$

$\Rightarrow 3^n-1=\frac{2\times120}{3}$

$\Rightarrow 3^n-1=80$

$\Rightarrow 3^n=81$

$\Rightarrow 3^n=3^4$

$\Rightarrow n=4$ ans.

Q.2) If a,b,c are in G.P. & $a^{\frac{1}{x}}=b^{\frac{1}{y}}=c^{\frac{1}{z}}$. Prove that x,y,z are in A.P.

Sol.2) Given: a,b,c are in G.P.

$\Rightarrow b^2=ac$

Given: $a^{\frac{1}{x}}=b^{\frac{1}{y}}=c^{\frac{1}{z}}$

Let $a^{\frac{1}{x}}=b^{\frac{1}{y}}=c^{\frac{1}{z}}=k$

$\Rightarrow a^{\frac{1}{x}}=k$

$\Rightarrow a=k^x$

$\Rightarrow b^{\frac{1}{y}}=k$

$\Rightarrow b=k^y$

$\Rightarrow c^{\frac{1}{z}}=k$

$\Rightarrow c=k^z$

We have, $b^2=ac$

$\Rightarrow (k^y)^2=(k^x).(k^z)$

$\Rightarrow k^{2y}=k^{x+z}$

$\Rightarrow 2y=x+z$

$\therefore x,y,z$ are in G.P. ans.

Q.3) Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the G.M. between a and b.

Sol.3) We have, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=G.M.=\sqrt{ab}$

$\Rightarrow \frac{a^{n+1}+b^{n+1}}{a^n+b^n}=a^{\frac{1}{2}}.b^{\frac{1}{2}}$

$\Rightarrow a^{n+1}+b^{n+1}=a^{\frac{1}{2}}.b^{\frac{1}{2}}(a^n+b^n)$

$\Rightarrow a^{n+1}+b^{n+1}=a^{n+\frac{1}{2}}.b^{\frac{1}{2}}+b^{n+\frac{1}{2}}.a^{\frac{1}{2}}$

$\Rightarrow a^{n+1}-a^{n+\frac{1}{2}}.b^{\frac{1}{2}}=b^{n+\frac{1}{2}}.a^{\frac{1}{2}}-b^{n+\frac{1}{2}}$

	$\Rightarrow a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)-b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)$ $\Rightarrow a^{n+\frac{1}{2}}=b^{n+\frac{1}{2}}$ $\Rightarrow \left(\frac{a}{b}\right)^{n+\frac{1}{2}}=1$ $\Rightarrow \left(\frac{a}{b}\right)^{n+\frac{1}{2}}=\left(\frac{a}{b}\right)^{0}$ $\Rightarrow n+\frac{1}{2}=0$ $\Rightarrow n=\frac{-1}{2}$ ans.
Q.4)	Insert three numbers between 3 and 243 so that the resulting sequence is an G.P.
Sol.4)	Here, $a=3\,,b=243\ and\ n=3$ Let G.M.S. are G_1, G_2, G_3 $\Rightarrow r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}=\left(\frac{243}{3}\right)^{\frac{1}{3+1}}=(81)^{\frac{1}{4}}=3$ $\therefore r=3$ Now, $G_1=ar^1=(3)(3)=9$ $G_2=ar^2=(3)(9)=27$ $G_3=ar^3=(3)(27)=81$ $\therefore$ required no.s are $3, 27, 81$ ans.
Q.5)	If the first and the n^{th} term of a G.P. are $a\ and\ b$ respectively and if P is the product of n terms. Prove that $P^2=(ab)^n$.
Sol.5)	Given, $a_1=a$ $\Rightarrow a_n=b$ $\Rightarrow ar^{n-1}=b$ $\Rightarrow r^{n-1}=\frac{b}{a}$ $\Rightarrow r=\left(\frac{b}{a}\right)^{\frac{1}{n-1}}$ Now, $P\rightarrow$ product of n terms $\Rightarrow P=a.ar.ar^2.ar^3.\dots\dots\dots..ar^{n-1}$ $\Rightarrow P=a^n.r^{1+2+3\dots\dots(n-1)}$ $\Rightarrow P=a^n.r^{\frac{n(n-1)}{2}}$ Putting the value of r $\Rightarrow P=a^n\left[\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right]^{\frac{n(n-1)}{2}}$ $\Rightarrow P=a^n\left(\frac{b}{a}\right)^{\frac{n}{2}}$ $\Rightarrow P=a^n.\frac{b^{\frac{n}{2}}}{a^{\frac{n}{2}}}$ $\Rightarrow P=a^{n-\frac{n}{2}}.b^{\frac{n}{2}}$ $\Rightarrow P=a^{\frac{n}{2}}.b^{\frac{n}{2}}$ $\Rightarrow P=(ab)^{\frac{n}{2}}$ Squaring $P^2=(ab)^n$ (proved)
Q.6)	If the $p^{th}, q^{th}\ and\ r^{th}$ terms of a G.P. are $a, b\ and\ c$ respectively.

	Prove that $a^{q-r}.b^{r-p}.c^{p-q}=1$	
Sol.6)	Let 1^{st} term $=A$ Common ratio $=R$ Given, $a_p=a=AR^{p-1}$ $a_q=b=AR^{q-1}$ $a_r=c=AR^{r-1}$ Taking L.H.S $a^{q-r}.b^{r-p}.c^{p-q}$ Substitute the value of a,b,c in L.H.S. = $[AR^{p-1}]^{q-r}.[AR^{q-1}]^{r-p}.[AR^{r-1}]^{p-q}$ = $A^{q-r}.R^{(p-1)(q-r)}.A^{r-p}.R^{(q-1)(r-p)}.A^{p-q}.R^{(r-1)(p-q)}$ = $A^{q-r+r-p+p-q}.R^{pq-pr-q+r+qr-pq-r+p+rq-rq-p+q}$ = $A^0.R^0$ = $(1)(1)$ = 1 R.H.S. (proved)	
Q.7)	If A and G be A.M. and G.M. respectively between two +ve numbers. Prove that he numbers are $A\pm\sqrt{(A+G)(A-G)}$.	
Sol.7)	Let the numbers are a and b Then $A=\frac{a+b}{2}$ $and\ G=\sqrt{ab}$ Consider, $A+\sqrt{(A+G)(A-G)}$ = $A+\sqrt{A^2-G^2}$ Put value of A & G = $\frac{a+b}{2}+\sqrt{\left(\frac{a+b}{2}\right)^2-\left(\sqrt{ab}\right)^2}$ = $\frac{a+b}{2}+\sqrt{\frac{a^2+b^2+2ab}{4}-ab}$ = $\frac{a+b}{2}+\sqrt{\frac{a^2+b^2-2ab}{4}}$ = $\frac{a+b}{2}+\sqrt{\left(\frac{a-b}{2}\right)^2}$ = $\frac{a+b}{2}+\frac{a-b}{2}$ = $2a$ $\therefore A+\sqrt{(A+G)(A-G)}=a$ Similarly, $A+\sqrt{(A+G)(A-G)}=b$ $\therefore$ The numbers are $A\pm\sqrt{(A+G)(A-G)}$ ans.	
Q.8)	Let s be the sum, P be the product and R be the sum of reciprocal of n terms in G.P. Prove that $P^2R^n=S^n$.	
Sol.8)	$S=a+ar+ar^2+\ldots\ldots ar^{n-1}$ $\Rightarrow S=a\left(\frac{r^n-1}{r-1}\right);r>1$ $P=a+ar+ar^2+\ldots\ldots ar^{n-1}$ $P=a^n+r^{1+2+\cdots\ldots(n-1)}$ $P=a^n.r^{\frac{n(n-1)}{2}}$ $R=\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\ldots\ldots\frac{1}{ar^{n-1}}$ It is also a G.P. with 1^{st} term $\frac{1}{a}$ and common ratio $\frac{1}{r}$ $\left(\because r>1\ \therefore\frac{1}{r}<1\right)$	

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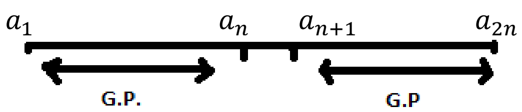
	$\therefore \mathrm{R} = \frac{1}{a}\left[\frac{1-\frac{1}{r^n}}{1-\frac{1}{r}}\right]$ $R = \frac{1}{a}\left[\frac{r^n-1}{r-1}\right].\frac{r}{r^n}$ Taking L.H.S. $P^2.R^n$ Put value of P and R in L.H.S. $\therefore$ L.H.S $\left[a^n.r^{\frac{n(n-1)}{2}}\right]\left[\frac{1}{a}\left(\frac{r^n-1}{r-1}\right).\frac{r}{r^n}\right]^n$ $= a^{2n}.r^{n(n-1)}.\frac{1}{a^n}\left(\frac{r^n-1}{r-1}\right)^n.\frac{r}{r^{n^2}}$ $= a^{2n-n}.r^{n^2-n+n-n^2}.\left(\frac{r^n-1}{r-1}\right)^n$ $= a^n.r^0.\left(\frac{r^n-1}{r-1}\right)^n$ $= a^n.\left(\frac{r^n-1}{r-1}\right)^n$ $= \left[a.\left(\frac{r^n-1}{r-1}\right)^n\right]$ $= S^n$ RHS ans.	
Q.9)	Show that the ratio of the sum of 1^{st} n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.	
Sol.9)	Here G.P. consist of $(2n)$ no. of terms  1^{st} terms $= a_1 = a$ 1^{st} term $= a_{n+1} = ar^n$ Ratio $= r$ Ratio$= r$ Terms $= n$ Terms$= n$ Sum$= S_n$ Sum$= S'_n$ S_n →sum of 1^{st} n terms S'_n →sym of terms from $(n+1)^{th}$ to $(2n)$ terms $S_n = a\left(\frac{r^n-1}{r-1}\right)$ $S'_n = ar^n\left[\frac{r^n-1}{r-1}\right]$ Now, $\frac{S_n}{S'_n} = \frac{a\left(\frac{r^n-1}{r-1}\right)}{ar^n\left(\frac{r^n-1}{r-1}\right)} = \frac{1}{r^n}$ (proved)	
Q.10)	If a, b, c, d and p are real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$ then show that a, b, c & d are in G.P.	
Sol.10)	To show that, a, b, c & d are in G.P. We have to prove $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$ Given, $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \leq 0$ $\Rightarrow a^2p^2 + b^2p^2 + c^2p^2 - 2abp - 2bcp - 2cdp + b^2 + c^2 + d^2 \leq 0$ $\Rightarrow (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \leq 0$ $\Rightarrow (ap-b)^2 + (bp-c)^2 + (cp-d)^2 \leq 0$ But $(ap-b)^2 + (bp-c)^2 + (cp-d)^2$ cannot less than 0 {$\because$ sum of square can never be negative}	

	$\therefore (ap-b)^2+(bp-c)^2+(cp-d)^2=0$ This is possible only when $ap-b=0$, $ap=b$, $\frac{b}{a}=p$ | $bp-c=0$, $bp=c$, $\frac{c}{b}=p$ | $cp-d=0$, $cp=d$, $\frac{d}{c}=p$ $\Rightarrow \frac{b}{a}=\frac{c}{b}=\frac{d}{c}$ $\Rightarrow a,b,c,d$ are in G.P. ans.	
Q.11)	If p,q,r are in G.P. and the equations $px^2+2qx+r=0$ and $dx^2+2ex+f=0$ have a common root, then show that $\frac{d}{p},\frac{e}{q},\frac{f}{r}$ are in A.P.	
Sol.11)	Given, p,q,r are in G.P. $\therefore\ q^2=pr$ To prove, $\frac{d}{p},\frac{e}{q},\frac{f}{r}$ are in G.P. i.e., $\frac{2e}{q}=\frac{d}{p}+\frac{f}{r}$ consider the equation, $px^2+2qx+r=0$ by quadratic formula, $x=\frac{-2q\pm\sqrt{4q^2-4pr}}{2p}$ $\Rightarrow x=\frac{-2q\pm\sqrt{4pr-4pr}}{2p}$ .... $\{\because\ q^2=pr\}$ $\Rightarrow x=\frac{-2q}{2p}$ $\Rightarrow x=\frac{-q}{p}$ This is also the root of the equation $dx^2+2ex+f=0$ $\therefore d\left(\frac{-q}{p}\right)^2+2e\left(\frac{-q}{p}\right)+f=0$ $\Rightarrow \frac{dq^2}{p^2}-\frac{2eq}{p}+f=0$ $\Rightarrow \frac{dpr}{p^2}-\frac{2eq}{p}+f=0$............. $\{\because\ q^2=pr\}$ $\Rightarrow \frac{dr}{p}-\frac{2eq}{p}+f=0$ $\Rightarrow dr-2eq+fp=0$ $\Rightarrow 2eq=dr+fp$ Divide by q^2 $\Rightarrow \frac{2e}{q}=\frac{dr}{q^2}+\frac{fp}{q^2}$ $\Rightarrow \frac{2e}{q}=\frac{dr}{pr}+\frac{fp}{pr}$ $\Rightarrow \frac{2e}{q}=\frac{d}{p}+\frac{f}{r}$ $\therefore\ \frac{d}{p},\frac{e}{q},\frac{f}{r}$ are in A.P. (proved)	
Q.12)	Fid the sum of the products of the corresponding terms of the sequences 2,4,8,16,32 and 128,32,8,2,$\frac{1}{2}$.	
Sol.12)	1st sequence $2,4,8,16,32$ 2nd sequence $128,32,8,2,\frac{1}{2}$ New sequence (products of corresponding terms) $=256,128,64,32,16$ Now, $\frac{a_2}{a_1}=\frac{128}{256}=\frac{1}{2}$	

	$\frac{a_3}{a_2} = \frac{64}{128} = \frac{1}{2}$ Clearly it is a G.P. with $a = 256, r = \frac{1}{2}$ and no. of term $= 5$ $\therefore$ sum $= S_n = a\left(\frac{1-r^n}{1-r}\right)$ $= S_5 = 256\left(\frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}}\right)$ $= 256\left(\frac{1-\frac{1}{32}}{\frac{1}{2}}\right)$ $= 2 \times 256\left(\frac{31}{32}\right)$ $= 16 \times 31$ $= 496$ ans.	
Q.13)	Find four numbers forming a G.P. in which the third term is greater than the first term by 9 & the second term is greater than fourth term by 18.	
Sol.13)	Let the four numbers are a, ar, ar^2, ar^3 We have, $a_3 = a_1 + 9$ $\Rightarrow ar^2 = a + 9$ $\Rightarrow ar^2 - a = 9$ $\Rightarrow a(r^2 - 1) = 9$ …………… (i) And $a_2 = a_4 + 18$ $\Rightarrow ar = ar^3 + 18$ $\Rightarrow ar - ar^3 = 18$ $\Rightarrow ar(r^2 - 1) = 18$ $\Rightarrow -ar(r^2 - 1) = 18$ …………… (ii) Divide (ii) and (i) $\frac{-ar(r^2-1)}{a(r^2-1)} = \frac{18}{9}$ $\therefore -r = 2$ $\Rightarrow r = -2$ Put in (i) $9(4-1) = 9$ $\Rightarrow 3a = 9$ $\Rightarrow a = 3$ $\therefore$ the no.s are $3, -6, 12, 24$ ans.	
Q.14)	Evaluate $\sum_{k=1}^{11}(2+3^k)$	
Sol.14)	$\sum_{k=1}^{11}(2+3^k) = (2+3^1) + (2+3^2) + (2+3^3) +$……..$(2+3^{11})$ $= (2+2+2+\cdots \ldots .11\ terms) + (3^1 + 3^2 + 3^3 \ldots \ldots \ldots 3^{11})$ G.P. $a = 3, r = 3, n = 11$ $= 22 + 3\left(\frac{3^{11}-1}{3-1}\right)$ $= 22 + \frac{3^{12}-3}{2}$ $= \frac{44+3^{12}-3}{2}$ $= \frac{41+3^{12}}{2}$ ans.	
Q.15)	If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p-q), (q-r), (r-s)$ are also in G.P.	

Sol.15)	$\Rightarrow a_p = a + (p-1)d$ .... $\{\because a_p, a_q, a_r, a_s \text{terms of an A. P.}\}$ $\Rightarrow a_q = a + (q-1)d$ $\Rightarrow a_r = a + (r-1)d$ $\Rightarrow a_s = a + (s-1)d$ Given that a_p, a_q, a_r, a_s are in G.P. $\therefore \frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{a_s}{a_r}$ .... (i) Consider $\frac{a_q}{a_p} = \frac{a_r}{a_q}$ $\Rightarrow \frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{a_q - a_r}{a_p - a_q}$ .... $\left\{if\ \frac{a}{b} = \frac{c}{d}\ then\ \frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-a}\right\}$ $\Rightarrow \frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{[a+(q-1)d]-[a+(r-1)d]}{[a+(p-1)d]-[a+(q-1)d]}$ $\Rightarrow \frac{d(q-1)-d(r-1)d}{d(p-1)-d(q-1)d}$ $\therefore \frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{q-r}{p-q}$ .... (ii) Now, consider $\frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{a_r - a_s}{a_q - a_r}$ $\Rightarrow \frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{[a+(r-1)d]-[a+(s-1)d]}{[a+(q-1)d]-[a+(r-1)d]}$ $\Rightarrow \frac{d(r-1-s+1)}{d(q-1-r+1)}$ $\therefore \frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{r-s}{q-r}$ .... (iii) From (i), (ii) & (iii) $\Rightarrow \frac{q-r}{p-q} = \frac{r-q}{q-r}$ $\Rightarrow (q-r)^2 = (p-q)(r-s)$ $\Rightarrow (p-q), (q-r), (r-s)$ are in G.P.	
Q.16)	If the 4th , 10th and 16th term of a G.P. are x, y, z respectively. Prove that x, y, z are in G.P.	
Sol.16)	$\Rightarrow a_4 = x \Rightarrow ar^3 = x$ $\Rightarrow a_{10} = y \Rightarrow ar^9 = y$ $\Rightarrow a_{16} = z \Rightarrow ar^{15} = z$ To prove, x, y, z are in G.P. i.e., to prove $y^2 = xz$ L.H.S. $y^2 = (ar^9)^2 = a^2r^{18}$ R.H.S. $xz = (ar^3)(ar^{15}) = a^2r^{18}$ Clearly $y^2 = xz$ $\therefore\ x, y, z$ are in G.P.	