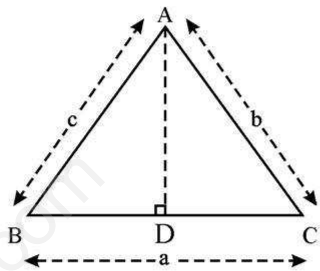
CHAPTER – 3

TRIGONOMETRIC FUNCTIONS

I. Law of Sines (or Sine Rule)

The sine rule states that the lengths of the sides of a triangle are proportional to the sines of angles opposite to them, i.e. in ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.



Proof : Three cases arise

Case (i) When ΔABC is acute angled triangle

Let $a = BC$, $b = AC$ and $c = AB$

From vertex A, draw $AD \perp BC$

In ΔABD, $\frac{AD}{AB} = \sin B \Rightarrow AD = c \sin B$ .....(i)

In ΔACD, $\frac{AD}{AC} = \sin C \Rightarrow AD = b \sin C$ ....(ii)

From (i) and (ii) we get, $c \sin B = b \sin C$

or $\frac{b}{\sin B} = \frac{c}{\sin C}$ .... (A)

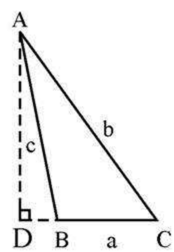
Similarly, by drawing $BE \perp AC$, we can prove that

$\frac{a}{\sin A} = \frac{c}{\sin C}$ .... (B)

From (A) and (B), we see that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Case (ii) When Δ ABC is an obtuse angled triangle



From vertex A, draw $AD \perp BC$ produced

In ΔABD, $\frac{AD}{AB} = \sin (180°-B)$

or $\frac{AD}{AB} = \sin B \Rightarrow AD = c \sin B$ .... (i)

Similarly, in ΔACD, $\frac{AD}{AC} = \sin C$

or $AD = b \sin C$ .... (ii)

From (i) and (ii), we get

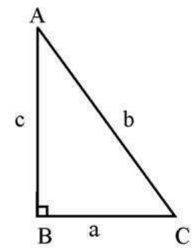
$c \sin B = b \sin C$ or $\frac{b}{\sin B} = \frac{c}{\sin C}$

Similarly, by drawing $BE \perp AC$, we can show that

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Case (iii) When ΔABC is a right angled triangle



In ΔABC, right angled at B

(i) $\frac{AB}{AC} = \sin C$ or $\frac{c}{b} = \sin C \Rightarrow b = \frac{c}{\sin C}$

(ii) $\frac{BC}{AC} = \sin A$ or $\frac{a}{b} = \sin A \Rightarrow b = \frac{a}{\sin A}$

(iii) $\sin B = \sin\frac{\pi}{2} = 1 \Rightarrow \frac{b}{\sin B} = b$

From (i), (ii) and (iii), we get

$$b = \frac{a}{\sin A} = \frac{c}{\sin c} = \frac{b}{\sin B}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\therefore$ From all the three cases, we see that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note : (i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

$\Rightarrow a = k \sin A,\ b = k \sin B,\ c = k \sin C$

(ii) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \lambda$ (say)

$\Rightarrow \sin A = a\lambda, \sin B = b\lambda, \sin C = c\lambda$

These can be used in solving problems

Example 1 : In ΔABC, if $a = 2$, $b = 3$ and $\sin A = \dfrac{2}{3}$, find $\angle B$.

Solution : We know that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Here $a = 2$, $b = 3$ and $\sin A = \dfrac{2}{3}$

$$\frac{2}{\frac{2}{3}} = \frac{3}{\sin B} \Rightarrow \sin B = 1 \Rightarrow B = \frac{\pi}{2} \text{ or } 90°$$

Example 2 : In any triangle ABC, if the angles are in the ratio of 1:2:3, prove that the corresponding sides are in the ratio of $1:\sqrt{3}:2$.

Solution : Let the angles be θ, 2θ and 3θ

As $\theta + 2\theta + 3\theta = 180° \Rightarrow \theta = 30°$

$\therefore$ The angles are 30°, 60°, 90°

Let the corresponding sides be a, b, c

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{a}{\sin 30°} = \frac{b}{\sin 60°} = \frac{c}{\sin 90°}$$

$$\text{or } \frac{a}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{1} = k \text{ (say)}$$

$$\Rightarrow a : b : c = \frac{k}{2} : \frac{\sqrt{3}}{2} k : k \text{ or } 1 : \sqrt{3} : 2$$

Example 3 : In any triangle, prove that

(i) $\dfrac{a^2 - c^2}{b^2} = \dfrac{\sin(A-C)}{\sin(A+C)}$ (ii) $b \cos B + c \cos C = a \cos(B-C)$

Solution : (i) LHS $= \dfrac{a^2 - c^2}{b^2} = \dfrac{k^2\sin^2 A - k^2\sin^2 C}{k^2\sin^2 B}$ [by sine formula]

$$= \frac{\sin^2 A - \sin^2 C}{\sin^2 B} = \frac{\sin (A+C) \cdot \sin (A - C)}{\sin^2 \left[180^\circ - (A+C)\right]}$$

$$= \frac{\sin (A+C) \cdot \sin (A-C)}{\sin (A+C) \cdot \sin (A+C)} = \frac{\sin (A-C)}{\sin (A+C)} = \text{RHS}$$

(ii) LHS $= b \cos B + c \cos C$

$= k [\sin B \cos B + \sin C \cos C]$

$$= \frac{k}{2} \left[\sin 2B + \sin 2C\right]$$

$$= \frac{k}{2} \left[2 \sin (B+C) \cos (B-C)\right]$$

$= k [\sin (180^\circ - A) \cos (B-C)]$

$= k \sin A \cos (B-C)$

$= a \cos (B-C) = \text{RHS}$

Example 4 : Prove that

$a (\sin B - \sin C) + b (\sin C - \text{Sin } A) + c (\sin A - \sin B) = 0$

Solution : We know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$ (say)

$\Rightarrow a = k \sin A,\ b = k \sin B,\ c = k \sin C$

$\therefore$ LHS $= k \sin A (\sin B - \sin C) + k \sin B (\sin C - \sin A) + k \sin C (\sin A - \sin B)$

$$= k \left[\sin \cancel{A} \cancel{\sin} B - \sin \cancel{A} \cancel{\sin} C + \sin \cancel{B} \cancel{\sin} C - \sin \cancel{B} \cancel{\sin} A + \sin \cancel{C} \cancel{\sin} A - \sin \cancel{C} \cancel{\sin} B\right]$$

$= k \cdot 0 = 0 = \text{RHS}$

Example 5 : In any ΔABC. prove that

$$\frac{a^2 + b^2}{a^2 + c^2} = \frac{1 + \cos (A-B) \cos C}{1 + \cos (A-C) \cos B}$$

Solution : We know that in ΔABC, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

$$\therefore \text{LHS} = \frac{\cancel{k^2} (\sin^2 A + \sin^2 B)}{\cancel{k^2} (\sin^2 A + \sin^2 C)} = \frac{1 - \cos^2 A + \sin^2 B}{1 - \cos^2 A + \sin^2 C} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)}$$

$$= \frac{1 - \cos (A-B) \cos (A+B)}{1 - \cos (A-C) \cos (A+C)}$$

$$\left\{ \begin{aligned} & \cos^2 A - \sin^2 B \\ & = \frac{1}{2} \left[\cancel{1} + \cos 2A - \cancel{1} + \cos 2B\right] \\ & = \frac{1}{\cancel{2}} \left[\cancel{2} \cos (A+B) \cos (A-B)\right] \end{aligned} \right\}$$

$$= \frac{1 + \cos(A\text{-}B)\cos C}{1 + \cos(A\text{-}C)\cos C} = \text{RHS}$$

EXERCISE 1

1. In Δ ABC, if a = 18, b = 24 and c = 30 and $\angle C = 90°$, find sin A, sin B and sin C.

2. In any Δ ABC, prove that

$$\frac{a+b}{c} = \frac{\cos\frac{(A\text{-}B)}{2}}{\sin\frac{C}{2}}$$

3. $\dfrac{b^2 - c^2}{a^2} = \dfrac{\sin(B - C)}{\sin(B + C)}$

4. If a cos A = b cos B, then the triangle is either isosceles or right angled.

5. $\dfrac{a - b}{a + c} = \dfrac{\tan\frac{(A\text{-}B)}{2}}{\tan\frac{(A+B)}{2}}$

6. $\sin\left(\dfrac{B - C}{2}\right) = \dfrac{b - c}{a}\cos\dfrac{A}{2}$

7. $\dfrac{1 + \cos(A\text{-}B)\cos C}{1 + \cos(A\text{-}C)\cos B} = \dfrac{a^2 + b^2}{a^2 - c^2}$

8. $(b - c)\cot\dfrac{A}{2} + (c - a)\cot\dfrac{B}{2} + (a - b)\cot\dfrac{C}{2} = 0$

9. $a\cos\left(\dfrac{B - C}{2}\right) = (b + c)\sin\dfrac{A}{2}$

10. $\dfrac{c}{a - b} = \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$

11. $a (\cos C - \cos B) = 2 (b - c) \cos^2 \frac{A}{2}$

12. $a \sin A - b \sin B = c \sin (A - B)$

II. Cosine Rule

In any triangle ABC, we have

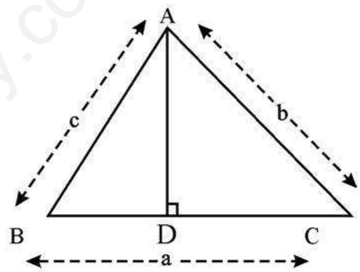
(i) $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

(ii) $b^2 = a^2 + c^2 - 2ac \cos B$ or $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

(ii) $c^2 = a^2 + b^2 - 2ab \cos C$ or $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Proof : Three cases arise :

Case I : When the ΔABC is an acute angled triangle.



From vertex A, draw $AD \perp BC$

In ΔABD, $\cos B = \frac{BD}{c} \Rightarrow BD = c \cos B$

In ΔACD, $\cos C = \frac{CD}{b} \Rightarrow CD = b \cos C$

Also, $AC^2 = CD^2 + AD^2$

$= AD^2 + (BC - BD)^2$

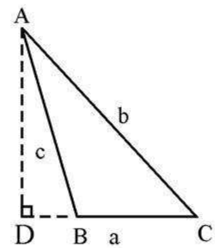
$= BC^2 + (AD^2 + BD^2) - 2BC.BD$

$AC^2 = BC^2 + AB^2 - 2BC.BD$

or, $b^2 = a^2 + c^2 - 2a.c \cos B$

or, $\cos B = \frac{a^2 + c^2 - b^2}{2\ ac}$

Case II : When ΔABC is an obtuse angled triangle.



From vertex A, draw $AD \perp CB$ produced

In ΔABD,

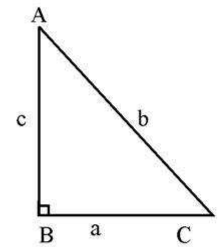
$\frac{BD}{c} = \cos (180^0 - B) = - \cos B$

$\Rightarrow BD = -c \cos B$

Also, $AC^2 = AD^2 + CD^2$

$$= AD^2 + (BC + BD)^2$$
$$= AD^2 + BD^2 + BC^2 + 2BC.BD$$
$$AC^2 = AB^2 + BC^2 + 2BC.BD$$
$$\text{or } b^2 = c^2 + a^2 + 2a(-c \cos B)$$
$$\text{or } \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Case III : When ΔABC is a right triangle.



$$b^2 = c^2 + a^2$$
$$\text{As } B = \frac{\pi}{2} \Rightarrow \cos B = 0$$
$$\therefore b^2 = c^2 + a^2 - 2ac \cos B \qquad [\because \cos B = 0]$$
$$\Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Thus, in all the three cases $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$

By following the same procedure, we can prove that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Let us now take some examples :

Example 6: In a ΔABC, prove that $a(b \cos C - c \cos B) = b^2 - c^2$

Solution : LHS : $a(b \cos C - c \cos B)$

$$= ab\left[\frac{a^2 + b^2 - c^2}{2ab}\right] - ac\left[\frac{a^2 + c^2 - b^2}{2ac}\right]$$
$$= \frac{1}{2}\left[\cancel{a^2} + b^2 - c^2 - \cancel{a^2} - c^2 + b^2\right]$$
$$= \frac{1}{2}\left[2b^2 - 2c^2\right] = b^2 - c^2 = RHS$$

Example 7 : If in any ΔABC, $\dfrac{b+c}{12} = \dfrac{c+a}{13} = \dfrac{a+b}{15}$, then prove that $\dfrac{\cos A}{2} = \dfrac{\cos B}{7} = \dfrac{\cos C}{11}$

Solution : $\frac{b+c}{12} = \frac{c+a}{13} = \frac{a+b}{15} = k$

$\Rightarrow b+c = 12k,\ c+a = 13k,\ a+b = 15k$

$(b+c) + (c+a) + (a+b) = 40k$

$\Rightarrow a+b+c = 20k$

$\Rightarrow a = 8k,\ b = 7k\ ,\ c = 5k$

$$\text{Cos } A = \frac{b^2+c^2-a^2}{2bc} = \frac{49k^2+25k^2-64k^2}{70k^2} = \frac{1}{7}$$

$$\text{Cos } B = \frac{a^2+c^2-b^2}{2ac} = \frac{64k^2+25k^2-49k^2}{80k^2} = \frac{1}{2}$$

$$\text{Cos } C = \frac{a^2+b^2-c^2}{2ab} = \frac{64k^2+49k^2-25k^2}{112k^2} = \frac{88}{112} = \frac{11}{14}$$

$$\text{Cos } A : \text{Cos } B : \text{Cos } C = \frac{1}{7} : \frac{1}{2} : \frac{11}{14} = 2 : 7 : 11$$

$$\therefore \frac{\text{Cos}A}{2} = \frac{\text{Cos}B}{7} = \frac{\text{Cos}C}{11}$$

Examples 8: In a Δ ABC , prove that $\frac{c-b\cos A}{b-c\cos A} = \frac{\cos B}{\cos C}$

Solution : LHS $= \frac{c-b\cos A}{b-c\cos A}$

$$= \frac{c-b\frac{(b^2+c^2-a^2)}{2bc}}{b-c\frac{(b^2+c^2-a^2)}{2bc}} = \frac{b}{c}\left[\frac{c^2+a^2-b^2}{b^2+a^2-c^2}\right]$$

$$\text{RHS} = \frac{\cos B}{\cos C} = \frac{\cancel{2}\cancel{a}b\,(a^2+c^2-b^2)}{\cancel{2}\cancel{a}c\,(a^2+b^2-c^2)} = \frac{b}{c}\left[\frac{c^2+a^2-b^2}{a^2+b^2-c^2}\right]$$

= LHS

Example 9 : In a Δ ABC , if a = 18, b = 24 and c = 30, find cosA, cosB and cosC

Solution : Here a = 18, b = 24 and c = 30

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{576 + 900 - 324}{1440}$$

$$= \frac{1152}{1440} = \frac{4}{5}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{324 + 900 + 576}{1080} = \frac{648}{1080} = \frac{3}{5}$$

$$\cos C = \frac{a^2 + c^2 - b^2}{2ac} = \frac{324 + 576 - 900}{864} = 0$$

$$\therefore \cos A = \frac{4}{5}, \cos B = \frac{3}{5}, \cos C = 0$$

Example 10 : In any Δ ABC, prove that

$$2\,(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$$

Solution : LHS $= 2\left\{bc\,\frac{(b^2 + c^2 - a^2)}{2bc} + ca\,\frac{a^2 + c^2 - b^2}{2ac} + ab\,\frac{a^2 + b^2 - c^2}{2ab}\right\}$

$$= b^2 + c^2 - \cancel{a^2} + \cancel{a^2} + \cancel{c^2} - \cancel{b^2} + a^2 + \cancel{b^2} - \cancel{c^2}$$

$$= a^2 + b^2 + c^2 = \text{RHS}$$

Example 11 : In any ΔABC, prove that

$$\frac{b^2 - c^2}{a^2}.\sin 2A + \frac{c^2 - a^2}{b^2}.\sin 2B + \frac{a^2 - b^2}{c^2}.\sin 2C = 0$$

$$\text{LHS} = \frac{b^2 - c^2}{a^2}.2\,(ka)\left[\frac{b^2 + c^2 - a^2}{2bc}\right] + \frac{c^2 - a^2}{b^2}.2\,(kb)\left[\frac{a^2 + c^2 - b^2}{2ac}\right] + \frac{a^2 - b^2}{c^2}.2\,(kc)\left[\frac{a^2 + b^2 - c^2}{2ab}\right]$$

$$= \frac{k}{abc}\left[(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) + (c^2 - a^2)(c^2 + a^2) - b^2(c^2 - a^2) + (a^2 + b^2)(a^2 - b^2) - c^2(a^2 - b^2)\right]$$

$$= \frac{k}{abc}\left[\cancel{b^4} - \cancel{c^4} - \cancel{a^2b^2} + \cancel{a^2}c^2 + \cancel{c^4} - \cancel{a^4} - \cancel{b^2c^2} + \cancel{a^2b^2} + \cancel{a^4} - \cancel{b^4} - \cancel{a^2}c^2 + \cancel{b^2c^2}\right] = 0 = \text{RHS}$$

Example 12. In any ΔABC, prove that $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

Solution : LHS = $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

$$=\frac{b^2+c^2-a^2}{2abc}+\frac{a^2+c^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}$$

$$=\frac{1}{2abc}\left[b^2+c^2-\cancel{a^2}+a^2+\cancel{c^2}-\cancel{b^2}+\cancel{a^2}+\cancel{b^2}-\cancel{c^2}\right]=\frac{a^2+b^2+c^2}{2abc}=\text{RHS}$$

EXERCISE 2

1. In a ΔABC, if a=3, b=5 and c=7, find cosA, cosB and cosC.
2. If the sides of a ΔABC are a=4, b=6 and c=8, show that 6 cosC = 4+3 cosB
3. In any ΔABC, prove that $a^2=(b+c)^2-4bc\cos^2\frac{A}{2}$
4. In a ΔABC, if $\angle B=60^\circ$, prove that (a+b+c) (a-b+c) = 3ac
5. In any ΔABC, prove that

   $$\left(b^2-c^2\right)\cot A+\left(c^2-a^2\right)\cot B+\left(a^2-b^2\right)\cot C=0$$

6. In a ΔABC if $\cos C=\frac{\sin A}{2\sin B}$, prove that the triangle is isosceles.
7. In any ΔABC, prove that

   $$(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}=c^2$$

8. In any triangle ABC, prove that $2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$
9. In any ΔABC, prove that

   $$\left(c^2+b^2-a^2\right)\tan A=\left(a^2+c^2-b^2\right)\tan B=\left(a^2+b^2-c^2\right)\tan C$$

10. In any ΔABC, if $\angle C=60^\circ$, prove that

$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{1}{a+b+c}$$

ANSWERS:

EXERCISE : 1

1. $\sin A = \frac{3}{5}$, $\sin B = \frac{4}{5}$, $\sin C = 1$

EXERCISE : 2

Ans.1. $\cos A = \frac{13}{14}$, $\cos B = \frac{11}{14}$, $\cos C = \frac{-1}{2}$

CHAPTER – 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS.

I. SQUARE - ROOT OF A COMPLEX NUMBER

We know that every negative real number has exactly two square - roots. For example:-

$\sqrt{-2} = \pm\sqrt{2}\,i$, $\sqrt{-6} = \pm\sqrt{6}\,i$ etc.

Let us now try to find the square - root of a complex number suppose, we have to find $\sqrt{a+bi}$

Let $= \sqrt{a+bi} = x + yi$.

Squaring we get

$a+bi = (x^2-y^2) + 2ixy$

Equating real and imaginary parts, we get

(i) $x^2-y^2 = a$, and (ii) $2xy = b \Rightarrow xy = b/2$

Now $(x^2+y^2)^2 = \left((x^2-y^2)^2 + (2xy)^2\right) = a^2 + b^2$

$\Rightarrow$ (iii) $x^2+y^2 = \sqrt{a^2+b^2}$ [Positive sign as LHS is always positive

From (i) and (iii), we can find x and y [using (ii)]

Let us consider some examples

Example1 : find the square - root of 3+4i

Solution : Let $\sqrt{3+4i} = x+yi$

$\therefore$ $(x+yi)^2 = 3+4i$ or $x^2-y^2=3$ and $2xy = 4$ (i)

Now, $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 3^2+4^2 = 25$

$\therefore$ $x^2+y^2 = 5$ (ii)

$^2+y^2) + (x^2-y^2) = 8 \Rightarrow x = \pm 2$ and $y^2 = 1 \Rightarrow y = \pm 1$

As xy is positive $\Rightarrow$ when x = 2, y = 1 and when x = -2, y = -1

$\Rightarrow$ The two square roots of 3 + 4i are

2+i and -2-i

Example 2: Find the square - root of -15 + 8i

Solution : Let $\sqrt{-15+8i} = x+yi$ _ _ _ _(i)

Squaring (i), we get $-15+8i = (x^2-y^2) + 2xyi$

Equating real and imaginary parts, we get

$x^2-y^2 = -15$ and $2xy = 8$

Now $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = (-15)^2 + (8)^2$

$= 225+64 = 289 = (\pm 17)^2$

$\therefore\ x^2+y^2 = 17$ (Rejecting negative sign)

We have found $x^2- y^2 = -15$

$\therefore\ 2x^2 = 2 \Rightarrow x = \pm 1$

and $2y^2 = 32 \Rightarrow y = \pm 4$

As x.y is positive $\Rightarrow$ x+yi has values

1+4i and - 1- 4i:

Example 3:

Find $\sqrt{5-12\,i}$

Solution: Let $\sqrt{5-12\,i} = x + yi$

$\Rightarrow$ 5-12i = $(x^2 - y^2) + 2xy\,i$

Equating real and imaginary parts, we get

(i) $x^2-y^2 = 5$ and (ii) $2xy = -12$

Now $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 5^2+12^2 = 169$

$\therefore\ x^2 + y^2 = 13$ _ _ _ _ (iii)

From (i), (ii) and (iii), we get

$2x^2 = 18 \Rightarrow x = \pm 3$

and $y = \pm 2$

As xy is negative $\Rightarrow$ when $x = 3$, $y = -2$

and when $x = -3$, $y = 2$

The required square - roots are 3-2 i and -3 + 2i

or $\pm$ (3- 2i)

EXERCISE 1

Find the square - roots of following complex numbers

(i) -15 - 8i

(ii) -3 - 4i

(iii) $2-2\sqrt{3}\ i$

(iv) $8 + 6i$

(v) $7 - 24i$

Examples4: Solve the following quadratic equation:

$$2x^2 + \sqrt{15}\, i\, x - i = 0$$

Solution : $2x^2 + \sqrt{15}\, i\, x - i = 0$

Here $b^2 - 4ac = \left(\sqrt{15}\, i\right)^2 + 4.\, 2\, i$

$= - 15 + 8\, i$

$$\Rightarrow x = \frac{-\sqrt{15}\, i \pm \sqrt{-15+8i}}{4}$$

Let $\sqrt{-15+8i} = a+bi$

$\Rightarrow a^2-b^2 = -15$ ____(i) and $2ab = 8$ ____(ii)

$(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2 = (-15)^2 + (8)^2 = 289$

$\Rightarrow a^2+b^2 = 17$ ____(iii)

$\therefore\ a^2 = 1 \Rightarrow a = \pm 1, b = \pm 4$

When $a = 1, b = 4$

When $a = -1,\ b = -4$

$\therefore\ a + ib = 1 + 4i$ or $-1-4i$

$$\therefore\ x = \frac{-\sqrt{15}\, i \pm (1+4i)}{4}$$

$$\therefore\ x = \frac{1 + \left(4 - \sqrt{15}\right)i}{4} \text{ or } \frac{-1 - \left(\sqrt{15}+4\right)i}{4}$$

Example 5 : Solve the quadratic equation

$x^2 - x + (1+i) = 0$

Solution: Here $a = 1, b = -1, c = 1+i$

Discriminant $= b^2 - 4ac = (-1)^2 - 4\,(1+i)$

$= 1 - 4 - 4i$

$= (1)^2 + (2i)^2 - 2.(1)\,(2i)$

$= (2i - 1)^2$

$$\Rightarrow \sqrt{b^2\text{-}4ac} = \pm\ (2i\text{-}1)$$

$$\therefore\ x = \frac{1 \pm (2i\text{-}1)}{2} = \text{-}i + 1,\ i$$

EXERCISE 2

Solve the following quadratic equations:

(i) $i\,x^2 - x + 12i = 0$

(ii) $x^2 - (3\sqrt{2} - 2i)x - \sqrt{2}\ i = 0$

(iii) $x^2 - (\sqrt{2} + i)x + \sqrt{2}\ i = 0$

(iv) $2x^2 - (3+7i)\ x + (9i - 3) = 0$

(v) $x^2 - (3\sqrt{2} - 2i)x + 6\sqrt{2}\ i = 0$

ANSWERS:

EXERCISE 1:

(i) $\pm(1\text{-}4i)$

(ii) $\pm(1\text{-}2i)$

(iii) $\pm(\sqrt{3} - i)$

(iv) $\pm(3 + i)$

(v) $\pm(4 - 3i)$

EXERCISE 2

(i) $-4i,\ 3i$

(ii) $\left(\frac{3\sqrt{2} - 2i}{2}\right) \pm \left(\frac{4 - \sqrt{2}\ i}{2}\right)$

(iii) $\sqrt{2},\ i$

(iv) $\frac{3+i}{2},\ 3i$

(v) $3\sqrt{2},\ 2i$

CHAPTER - 9

SEQUENCES AND SERIES

I. Sum to infinity of a G.P.

Let us consider the G.P.

$1, \frac{3}{5}, \frac{9}{25}, ----$

Here a = 1, r = $3/5$

$$\therefore S_n = \frac{\left[1-\left(3/5\right)^n\right]}{\left(1-3/5\right)} = \frac{5}{2}\left[1-\left(3/5\right)^n\right] \text{ --- (i)}$$

Let us study the behaviour of $\left(3/5\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{3}{5}\right)^n$	$3/5$	$\left(3/5\right)^5$	$\left(3/5\right)^{10}$	$\left(3/5\right)^{20}$
	= 0.6	= 0.07776.	= 0.006047	0.00003656

We observe that as n becomes larger and larger, $\left(\frac{3}{5}\right)^n$ becomes closer and closer to zero.

In other words, we can say that as $n \to \infty$, $\left(\frac{3}{5}\right)^n \to 0$

Thus, from (i) we find that the sum to infinitely many terms (S_∞) of the above geometric progression is given by $S_\infty = \frac{5}{2}$

Now, for a geometric progression a, ar, ar^2, ____ if $|r| < 1$, then

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

As $n \to \infty$, $r^n \to 0$, as $|r| < 1$

$$\therefore\ S_\infty \to \frac{a}{1 - r}$$

For example

(i) $$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \text{----} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \text{ as } r = \frac{1}{3}$$

(ii) $$1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \text{----} = \frac{1}{1 - \left(\frac{-1}{3}\right)} = \frac{3}{4}, \text{ as } r = -\frac{1}{3}$$

Example 1 : Find the sum of the infinite GP $1, \frac{2}{3}, \frac{4}{9}, \text{---}$

Solution : Here $a = 1$, $r = \frac{2}{3}$ i.e. $r < 1$

$$\therefore\ S_\infty = \frac{a}{1 - r} = \frac{1}{1 - \frac{2}{3}} = 3$$

Example 2 : Find the sum to infinity of the GP

10, -9, 8.1, -7.29, ---

Solution: Here $a = 10$, $r = -0.9$

Since $|r| < 1$

$$\therefore\ S_\infty = \frac{a}{1 - r}$$

$$= \frac{10}{1 - (-0.9)} = \frac{10}{1.9} = \frac{100}{19}$$

$= 5.263$

Example 3 : Find the sum to infinity of the series

$$\frac{1}{3}+\frac{1}{5^2}+\frac{1}{3^3}+\frac{1}{5^4}+\frac{1}{3^5}+\frac{1}{5^6}+ \text{------}$$

Solution: We here

$$\frac{1}{3}+\frac{1}{5^2}+\frac{1}{3^3}+\frac{1}{5^4}+\frac{1}{3^5}+\frac{1}{5^6}+\text{----}=\left[\frac{1}{3}+\frac{1}{3^3}+\frac{1}{3^5}+\text{----}\right]+\left[\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\text{----}\right]$$

$$=\frac{1/3}{1-\frac{1}{3^2}}+\frac{\frac{1}{5^2}}{1-\frac{1}{5^2}}=\frac{1}{3}\text{ x }\frac{9}{8}+\frac{1}{25}\text{ x }\frac{25}{24}$$

$$=\frac{3}{8}+\frac{1}{24}=\frac{10}{24}=\frac{5}{12}$$

Example 4 : Prove that $3^{1/2}\,.\,3^{1/4}\,.\,3^{1/8}\text{ -----}=3$

Solution : We have $3^{1/2}\,.\,3^{1/4}\,.\,3^{1/8}\text{ ----}$

$$=3^{\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{---}\right]}$$

$$=3^{\frac{\frac{1}{2}}{1-1/2}}=3^1=3$$

II. Recurring decimal numbers as geometric series.

The sum to infinity of a geometric progression, with $|r| < 1$, can be applied in the infinite recurring non:terminating decimal expansion of some real numbers. Let us take the simple case of $0.\overline{3}=0.3333\text{ ---}$

We can write $0.3333\text{ ----}=0.3+0.03+0.003+\text{ ----}$ (i)

The RHS of (i), is the sum of infinite GP with a = 0.3 and r = 0.1 ($|r|<1$).

$$\therefore\ S_\infty=\frac{0.3}{1-0.1}=\frac{0.3}{0.9}=\frac{1}{3}$$

Thus, $0.\overline{3}=\frac{1}{3}$ or we can say that the rational number $\frac{1}{3}$, when expressed as a decimal will have $0.\overline{3}$ as its expansion.

Example 5 : Find a rational number, which when expressed as a decimal, will have $0.6\overline{8}$ as its expansion.

Solution : We write

$$0.6\overline{8} = 0.68888----$$

$$= 0.6 + [0.08 + 0.008 + 0.0008 + \_\_\_]$$

$$= 0.6 + \frac{0.08}{1-0.1} = 0.6 + \frac{0.08}{0.9}$$

$$= \frac{6}{10} + \frac{8}{90} = \frac{54+8}{90} = \frac{62}{90} = \frac{31}{45}$$

Hence, the required rational number is $\frac{31}{45}$

Example 6: The first term of a G.P. is 2 and sum to infinity in 6. Find the common ratio.

Solution: Here a=2 $S_\infty = 6$

$$\therefore 6 = \frac{2}{1\text{-r}} \text{ or } 1\text{-r} = 2/6 = 1/3$$

$$\Rightarrow r = 1\text{-} 1/3 = \frac{2}{3}$$

Exercise 1.

Find the sum to infinity in each of the following geometric progressions:

1. $5, \frac{20}{7}, \frac{80}{49}, ---$
2. $6, 1.2, 0.24, \_\_\_$
3. $1, \frac{-1}{3}, \frac{1}{3^2}, -\frac{1}{3^3}, \frac{1}{3^4}, ---$
4. $\frac{-5}{4}, \frac{5}{16}, -\frac{5}{64}, ---$
5. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, ---$
6. $0.3, 0.18, 0.108, \_\_\_$

7. $(\sqrt{2}+1), 1, (\sqrt{2}-1), (\sqrt{2}-1)^2, \_\_\_$

8. The common ratio of a GP is $-4/5$ and the sum to infinity is $\frac{80}{9}$. Find the first term.

9. Find an infinite GP whose first term is 1 and each term is the sum of all the terms which follow it.

10. The sum of first two terms of an infinite GP is 5 and each term is three times the sum of the succeeding terms. Find the GP.

11. Find the rational number having the following decimal expansions:

(i) $0.1\overline{5}$ (ii) $0.\overline{712}$ (iii) $3.5\overline{2}$ (iv) $0.\overline{231}$ (v) $0.3\overline{56}$

12. Let $x = 1+a+a^2 + \_\_\_$ and $y = 1+b+b^2 \_\_\_$, where $|a| < 1$ and $|b| < 1$. Prove that

$$1+ab+a^2b^2+\_\_\_ = \frac{xy}{x+y-1}.$$

13. If the sum of an infinite geometric series is 15 and the sum of the squares of three terms is 45. Find the series.

14. The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, find the G.P.

15. Prove that $6^{\frac{1}{2}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{8}} \cdot \_\_\_ = 6$

ANSWERS:

EXERCISE : 1

1. $\frac{35}{3}$ 2. 7.5 3. $\frac{3}{4}$ 4. -1 5. $\frac{-3}{5}$

6. 0.75 7. $\frac{4+3\sqrt{2}}{2}$ 8. 16 9. $1, \frac{1}{2}, \frac{1}{4}, ---$

10. $4, 1, \frac{1}{4}, \frac{1}{16}, ---$ 11. (i) $\frac{114}{99}$ (ii) $\frac{712}{999}$ (iii) $\frac{317}{90}$ (iv) $\frac{231}{999}$ (v) $\frac{353}{990}$

12. $5+\frac{10}{3}+\frac{20}{9}+---$ 14. $19, \frac{38}{3}, \frac{76}{9}, ---$

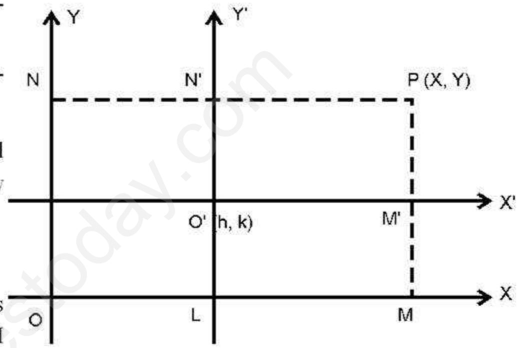
CHAPTER - 10

STRAIGHT LINE

I. SHIFTING OF ORIGIN

The position of origin and the direction of axes plays a major role in describing a curve in terms of equations. An equation corresponding to a set of points with reference to a system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system. One such transformation is when origin is shifted to a new point and new axes are transformed parallel to the original axes.

To see how the coordinates of a point of the plane changed under shifting of origin (or translation of axes).



Let O be the origin and P (x,y) be a point referred to the axes OX and OY.

Let O' X' and O' Y' be the new axes parallel to OX and OY respectively, where O' is the new origin

Let OL = h and O' L = k

Let the coordinates of P referred to new axes be (X, Y) then O' M' = X and P' M' = Y, OM = x and PM = y

$\therefore$ x = OM = OL +LM = h + O'M' = h+X

y = PM = MM'+PM' = K+PM' = k + Y

Thus, x = X + h, y = Y+k give the relation between the old and new coordinates.

Thus, if the equation of the set of points P with respect to OX and OY be f (x, y) = 0, the equation to the same set of points when origin is shifted to O' becomes f (X+h, Y+k) = 0, where X, Y are coordinates with reference to new axes O' X' and O' Y'.

If, therefore, the origin is shifted at a point (h,k), we should substitude X + h and Y + k for x and y respectively.

The transformation formula from new axes to old axes is X = x-h, Y = y-k. The coordinates of old origin, referred to new axes are (-h, -k).

Example1: Find the new coordinates of the point (3,-5) if origin is shifted to the point (2,3) by a translation of axes.

Solution : Coordinates of new origin are (h,k) = (2,3) and the original coordinates of point are (3,-5) = (x,y)

$\therefore$ new coordinates (X, Y) are given by

x = X + h, y = Y + k i.e. 3 = X + 2, -5 = Y + 3

$\therefore$ X = 3-2 = 1, Y = -5-3 = -8

Hence, the coordinates of the point (3,-5) in new system are (1, -8)

Example 2: Find what the equation $x^2 + xy - 3y^2 - y + 2 = 0$ becomes when the origin is shifted to the point (1,1)?

Solution : Let the coordinates of a point P changes from (x,y) to (X,Y) when origin is shifted to (1,1)

$\therefore x = X+1, y = Y+1$

Substiluting in the given equation, we get

$(X+1)^2 + (X+1)(Y+1) - 3(Y+1)^2 - (y+1) + 2 = 0$

$\Rightarrow x^2 + 2x + 1 + XY + X + Y + 1 - 3(y^2 + 2y + 1) - (Y + 1) + 2 = 0$

$\Rightarrow X^2 - 3Y^2 + XY + 3X - 6Y - 1 = 0$

$\therefore$ Equation in new system is $X^2 - 3Y^2 + XY + 3X - 6Y = 0$

Example 3: Find the point to which the origin should be shifted after shifting of origin so that the equation $x^2 - 12x + 4 = 0$ will have no first degree term.

Solution : Let origin be shifted to (h, k) and P (x, y) becomes

P (X + h, Y+k). Substituting in the given equation we get

$(X+h)^2 - 12(X+h) + 4 = 0$

$\Rightarrow X^2 + 2hX + h^2 - 12X - 12h + 4 = 0$

Since there is no first degree term : $2h - 12 = 0$

or $h = 6$

Hence origin should be shifted to (6,k) for any real value k.

Example 4: Verify that the area of the triangle with vertices (4,6), (7,10) and (1,-2) remains invariant under the translation of axes when origin is shifted to the point (-2, 1)

Solution : Let P (4, 6), Q (7,10) and R (-1,2) be the given points

$\therefore$ Area of Δ PQR $= \frac{1}{2}[4(10-2) + 7(2-6) - 1(6-10)]$

$= \frac{1}{2}[32 - 28 + 4] = 4$ sq.U.

Now shifting (x,y) to (X-2, Y+1)

New coordinates are $X = x+2, Y = y-1$

$\therefore$ P (4, 6) $\rightarrow$ (6, 5)

Q (7, 10) $\rightarrow$ (9, 9)

R (-1, 2) $\rightarrow$ (1, 1)

$\therefore$ Area of $\Delta = \frac{1}{2}$ [6 (9-1) + 9 (1-5) + 1 (5-9)]

$= \frac{1}{2}$ [48 - 36 -4] = 4 sq. units

Hence the area remains invarient.

Exercise 1.

1. Find the new coordinates of the points in each of the following, if the origin is shifted to the point (1,2) by translation of axes:

   (i) (4, 4) (ii) (4, 5)

   (iii) (9, 4) (iv) (3,2)

   (v) (7, -1) (vi) (2,5)

2. Find what the following equations become when origin is shifted to the point (2,3)

   (i) $x^2 + 2xy - y^2 + y + 3 = 0$ (ii) $3xy - x^2 - y + x = 0$

   (iii) $4xy + 2x - 3y + 2 = 0$ (iv) $x^2 + y^2 - 3x + 4y = 0$

3. If the origin is shifted to the point (1,-2), what do the following equations become?

   (i) $2x^2 + y^2 - 4x + 4y = 0$ (ii) $y^2 - 4x + 4y + 8 = 0$

4. At what point the origin be shifted, if the coordinates of a point (4,5) becomes (-3,9) ?

5. Prove that the area of a triangle is invariant under the translation of the axes.

ANSWERS:

1. (i) (3,2) (ii) (3,3) (iii) (8, 2) (iv) (2,0) (v) (6,-3) (vi) (1,3)

2. (i) $x^2 - y^2 + 2xy + 10x - y - 5 = 0$

   (ii) $3xy - x^2 + 6x + 5y + 13 = 0$

   (iii) $4xy + 14x + 5y + 21 = 0$

   (iv) $x^2 + y^2 + x + 10y + 19 = 0$

3. (i) $2x^2 + y^2 = 6$, (ii) $y^2 = 4x$

4. (7, -4)

CHAPTER – 13

LIMIT AND DERIVATIVES

I. SOME IMPORTANT LIMITS

1. $\lim\limits_{x \to 0} \dfrac{1}{x}$

We can easily observe that as $x \to 0$ from the left hand side, $\dfrac{1}{x}$, gets smaller and smaller i.e. $\dfrac{1}{x} \to -\infty$,

and as $x \to o$ from right hand side, $\dfrac{1}{x}$ gets greater and greater i.e. $\dfrac{1}{x} \to +\infty$

So, as x approaches to 0, either from left hand side or from right hand side, $\dfrac{1}{x}$ never approaches to a finite number.

Hence, we say that $\lim\limits_{x \to 0^-} \dfrac{1}{x}$ and $\lim\limits_{x \to 0^+} \dfrac{1}{x}$ both do not exist i.e. $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

2. $\lim\limits_{x \to \infty} \dfrac{1}{x}$

When x takes positive values only and successive values of x increase and become greater than any pre - assigned positive real number, however large it may be, then we say that x tends to infinity, i.e. $x \to \infty$

Clearly, as $x \to \infty$, $\dfrac{1}{x} \to 0$.

Thus we say that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

3. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

Let $x > 1$ then $\dfrac{1}{x} < 1$

$\therefore$ Using Binomial Theorem, we have

$$\left(1+\frac{1}{x}\right)^x = 1 + x\,\frac{1}{x} + \frac{x\,(x-1)}{2!}\left(\frac{1}{x}\right)^2 + \frac{x\,(x-1)(x-2)}{3!}\left(\frac{1}{x}\right)^3 + \_\,\_\,\_$$

$$1 + 1 + \left(\frac{1 - \frac{1}{x}}{2!}\right) + \left(\frac{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)}{3!}\right) + \_\_\_$$

When $x \to \infty$, each one of $\frac{1}{x}, \frac{2}{x}, \frac{3}{x}$ ___ etc will tend to zero

$$\therefore \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \_\_\_ = e.$$

Thus, $\therefore \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

4. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$

Taking $\frac{1}{x} = y$, we have, as $x \to 0$, $y \to \infty$

$$\therefore \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y = e$$

Hence $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

5. $\lim_{x \to 0} \frac{\log (1 + x)}{x}$

$$\lim_{x \to 0} \frac{\log (1 + x)}{x} = \lim_{x \to 0} \frac{1}{x} \log(1 + x) = \lim_{x \to 0} \log (1 + x)^{\frac{1}{x}}$$

$= \log e = 1$ $\quad \{\log (1 + x)$ means $\log_e (1 + x)\}$

6. $\lim_{x \to 0} \frac{e^x - 1}{x}$

Taking $e^x - 1 = y$, we have, as $x \to 0$, $y \to 0$

and $e^x = 1 + y \Rightarrow x = \log (1+y)$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log(1+y)} = \lim_{y \to 0} \frac{1}{\frac{\log(1+y)}{y}} = \frac{1}{1} = 1$$

Hence $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

7. $\lim_{x \to 0} \frac{a^x - 1}{x}, a > 0$

Taking $a^x - 1 = y$, we have, as $x \to 0, y \to 0$ and $a^x = 1 + y \Rightarrow x \log a = \log(1 + y)$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{y \to 0} \frac{y \log a}{\log(1+y)} = \lim_{y \to 0} \frac{\log a}{\frac{\log(1+y)}{y}} = \log a$$

Hence, $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$