

Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

INTRODUCTION

To prove certain results or statements in Algebra, that are formulated in terms of n , where n is a natural number, we use a specific technique called principle of mathematical induction (P.M.I)

Steps of P.M.I

Step I - Let $p(n)$: result or statement formulated in terms of n (given question)

Step II – Prove that $P(1)$ is true

Step III – Assume that $P(k)$ is true

Step IV – Using step III prove that $P(k+1)$ is true

Step V - Thus $P(1)$ is true and $P(k+1)$ is true whenever $P(k)$ is true.

Hence by P.M.I, $P(n)$ is true for all natural numbers n

Type I

Eg: Ex 4.1

1) Prove that

$$1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$$

Solution:-

Step I : Let $P(n)$: $1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$

Step II: $P(1)$:

$$\text{LHS} = 1$$

$$\text{RHS} = \frac{3-1}{2} = \frac{2}{2} = 1$$

$$\text{LHS} = \text{RHS}$$

Therefore $p(1)$ is true.

Step III: Assume that $P(k)$ is true

$$\text{i.e } 1+3+3^2+\dots+3^{k-1} = \frac{3^k-1}{2} \quad \text{--- (1)}$$

Step IV: we have to prove that $P(k+1)$ is true.

$$\text{ie to prove that } 1+3+3^2+\dots+3^{k-1}+3 = \frac{3^{k+1}-1}{2}$$

Proof

$$\text{LHS} = (1+3+3^2+\dots+3^{k-1}) + 3$$

$$= \frac{3^k-1}{2} + 3^k \text{ from eq(1)}$$

$$= \frac{3^k-1 + 2 \cdot 3^k}{2}$$

$$= \frac{3 \cdot 3^k - 1}{2} = \frac{3^{k+1}-1}{2} = \text{RHS}$$

Therefore $P(k+1)$ is true

Step V: Thus $P(1)$ is true and $P(k+1)$ is true whenever $P(k)$ is true. Hence by

P.M.I, $P(n)$ is true for all natural number n .

Text book

Ex 4.1

Q. 1,2, 3**(HOT), 4, 5*,6*,7,8,9,10*,11**,12,13**,14**,15,16**,17**,
eg 1, eg 3

Type 2

Divisible / Multiple Questions like Q. 20**,21,22**,23 of Ex 4.1

eg 4, eg 6**(HOT)

Q 22. Prove that $3^{2n+2}-8n-9$ is divisible by 8 for all natural number n .

Solution

Step I: Let $p(n): 3^{2n+2}-8n-9$ is divisible by 8

Step II: $P(1): 3^4 - 8 - 9 = 81 - 17 = 64$ which is divisible by 8

Therefore $p(1)$ is true

Step III: Assume that $p(k)$ is true

$$\text{i.e } 3^{2k+2} - 8k - 9 = 8m; \quad m \text{ is a natural number.}$$

$$\text{i.e } 3^{2k} \cdot 9 = 8m + 8k + 9$$

$$\text{ie } 3^{2k} = \frac{8m + 8k + 9}{9} \quad \text{_____ (1)}$$

Step IV: To prove that $p(k+1)$ is true.

ie to prove that $3^{2k+4} - 8(k+1) - 9$ is divisible by 8.

$$\text{Proof: } 3^{2k+4} - 8k - 17 = 3^{2k} \cdot 3^4 - 8k - 17 = \left(\frac{8m+8k+9}{9}\right) \times 3^4 - 8k - 17 \text{ (from eqn (1))}$$

$$= (8m+8k+9)9 - 8k - 17 = 72m + 72k + 81 - 8k - 17 = 72m - 64k + 64 = 8[9m - 8k + 8] \text{ is divisible by 8.}$$

Step V: Thus $P(1)$ is true and $P(k+1)$ is true whenever $P(k)$ is true. hence by P.M.I, $P(n)$ is true for all natural numbers n .

Type III: Problems based on Inequations

Ex 4.1 Q. 18,14, eg 7

(Q 18) Prove that $1+2+3+\dots+n < \frac{(2n+1)^2}{8}$

Step I : Let $P(n): 1+2+3+\dots+n < \frac{(2n+1)^2}{8}$

Step II: $P(1): 1 < \frac{9}{8}$ which is true, therefore $p(1)$ is true.

Step III: Assume that $P(k)$ is true.

$$\text{ie } 1+2+3+\dots+k < \frac{(2k+1)^2}{8} \text{ (1)}$$

Step IV: We have to prove that $P(k+1)$ is true. ie to

$$\text{prove that } 1+2+3+\dots+k+(k+1) < \frac{(2k+3)^2}{8}$$

Proof: Adding $(k+1)$ on both sides of inequation (1)

$$1+2+3+\dots+k+(k+1) < \frac{(2k+1)^2}{8} + (k+1)$$

$$= \frac{(4k^2+4k+1)+8k+8}{8}$$

$$= \frac{4k^2+12k+9}{8}$$

$$= \frac{(2k+3)^2}{8}$$

$$\text{Therefore } 1+2+3+\dots+k+(k+1) < \frac{(2k+3)^2}{8}$$

$P(k+1)$ is true.

Step V: Thus $P(1)$ is true and $P(k+1)$ is true whenever $P(k)$ is true. Hence by P.M.I, $P(n)$ is true for all natural number n .

HOT/EXTRA QUESTIONS

Prove by mathematical induction that for all natural numbers n .

- 1) $a^{2n-1} - 1$ is divisible by $a-1$ (type II)
- 2) $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer (HOT)
- 3) $\sin x + \sin 3x + \dots + \sin (2n-1)x = \frac{\sin^2 nx}{\sin x}$ (HOT Type 1)
- 4) $3^{2n-1} + 3^n + 4$ is divisible by 2 (type II)
- 5) Let $P(n)$: $n^2 + n - 19$ is prime, state whether $P(4)$ is true or false
- 6) $2^{2n+3} \leq (n+3)!$ (type III)
- 7) What is the minimum value of natural number n for which $2^n < n!$ holds true?
- 8) $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25 (type II)

Answers

- 5) false
- 7) 4