

CHAPTER - 13**LIMITS AND DERIVATIVES****KEY POINTS**

- $\lim_{x \rightarrow c} f(x) = l$ if and only if

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$$

- $\lim_{x \rightarrow c} \alpha = \alpha$, where α is a fixed real number.
- $\lim_{x \rightarrow c} x^n = c^n$, for all $n \in \mathbb{N}$
- $\lim_{x \rightarrow c} f(x) = f(c)$, where $f(x)$ is a real polynomial in x .

Algebra of limits

Let f, g be two functions such that $\lim_{x \rightarrow c} f(x) = l$ and $\lim_{x \rightarrow c} g(x) = m$, then

- $\lim_{x \rightarrow c} [\alpha f(x)] = \alpha \lim_{x \rightarrow c} f(x)$
 $= \alpha l$ for all $\alpha \in \mathbb{R}$
- $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x) = l \pm m$
- $\lim_{x \rightarrow c} [f(x).g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = lm$
- $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{l}{m}$, $m \neq 0$ $g(x) \neq 0$

- $\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow c} f(x)} = \frac{1}{l}$ provided $l \neq 0$ $f(x) \neq 0$
- $\lim_{x \rightarrow c} [(f(x)]^n = \left[\left(\lim_{x \rightarrow c} f(x) \right) \right]^n = l^n$, for all $n \in \mathbb{N}$

Some important theorems on limits

- $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(-x)$
- $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ where x is measured in radians.
- $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = 1$
- $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$ [Note that $\lim_{x \rightarrow 0} \frac{\cos x}{x} \neq 1$]
- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$
- $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$
- $\lim_{x \rightarrow 0} \frac{\log(1 + x)}{x} = 1$
- $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$

- A function f is said to have a derivative at any point x if it is defined in some neighbourhood of the point x and $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exists.

The value of this limit is called the derivative of f at any point x and is denoted by $f'(x)$ i.e.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Algebra of derivatives :

- $\frac{d}{dx}(cf(x)) = c \cdot \frac{d}{dx}(f(x))$ where c is a constant
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
- $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \frac{d}{dx}(f(x))$
- $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\cdot\frac{d}{dx}(g(x))}{(g(x))^2}$$
- If $y = f(x)$ is a given curve then slope of the tangent to the curve at the point (h, k) is given by $\left.\frac{dy}{dx}\right|_{(h,k)}$ and is denoted by 'm'.

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

Evaluate the following Limits :

$$1. \lim_{x \rightarrow 3} \frac{\sqrt{2x+3}}{x+3}$$

$$2. \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

3. $\lim_{x \rightarrow 0} \frac{\tan^2 3x}{x^2}$

4. $\lim_{x \rightarrow 2} (x^2 - 5x + 1)$

Differentiate the following functions with respect to x :

5. $\frac{x}{2} + \frac{2}{x}$

6. $x^2 \tan x$

7. $\frac{x}{\sin x}$

8. $\log_x x$

9. 2^x

10. If $f(x) = x^2 - 5x + 7$, find $f'(3)$

11. If $y = \sin x + \tan x$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

12. If $f(x) = \begin{cases} 5x - 4, & 0 < x \leq 1, \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$ show that $\lim_{x \rightarrow 1} f(x)$ exists.

13. If $f(x) = \begin{cases} \frac{x - |x|}{x}, & x \neq 0, \\ 2, & x = 0 \end{cases}$, show that $\lim_{x \rightarrow 0} f(x)$ does not exist.

14. Let $f(x)$ be a function defined by

$$f(x) = \begin{cases} 4x - 5, & \text{If } x \leq 2, \\ x - \lambda, & \text{If } x > 2, \end{cases} \text{ Find } \lambda, \text{ if } \lim_{x \rightarrow 2} f(x) \text{ exists}$$

Evaluate the following limits :

$$15. \lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3}$$

$$16. \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$$

$$17. \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$$

$$18. \lim_{x \rightarrow a} \frac{\frac{5}{7}x - \frac{5}{7}a}{\frac{2}{7}x - \frac{2}{7}a}$$

$$19. \lim_{x \rightarrow a} \frac{(x+2)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x-a}$$

$$20. \lim_{x \rightarrow 0} \frac{1 - \cos 2mx}{1 - \cos 2nx}$$

$$21. \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$$

$$22. \lim_{x \rightarrow 0} \frac{x \tan x}{1 - \cos x}$$

$$23. \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$$

$$24. \lim_{x \rightarrow a} \frac{\cos x - \cos a}{\cot x - \cot a}$$

$$25. \lim_{x \rightarrow \pi} \frac{1 + \sec^3 x}{\tan^2 x}$$

$$26. \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$$

27.
$$\lim_{x \rightarrow 1} \frac{x-1}{\log_e x}$$

28.
$$\lim_{x \rightarrow e} \frac{\log x - 1}{x - e}$$

29.
$$\lim_{x \rightarrow 4} \frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$$

30.
$$\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$$

31.
$$\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x}$$

Differentiate the following functions with respect to x from first principle:

32.
$$\sqrt{2x+3}$$

33.
$$\frac{x^2 + 1}{x}$$

34.
$$e^x$$

35.
$$\log x$$

36.
$$\operatorname{cosec} x$$

37.
$$\cot x$$

38.
$$a^x$$

Differentiate the following functions with respect to x :

39.
$$\frac{(3x+1)(2\sqrt{x}-1)}{\sqrt{x}}$$

40.
$$\left(x - \frac{1}{\sqrt{x}}\right)^3$$

41.
$$\left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$$

42.
$$\frac{\sin x - x \cos x}{x \sin x + \cos x}$$

43.
$$x^3 e^x \sin x$$

45. $\frac{e^x + \log x}{\sin x}$

46. $\frac{1 + \log x}{1 - \log x}$

47. $e^x \sin x + x^n \cos x$

48. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, prove that $2x \frac{dy}{dx} + y = 2\sqrt{x}$

49. If $y = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$ find $\frac{dy}{dx}$

50. If $y = \sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}$, prove that

$$(2xy) \frac{dy}{dx} = \frac{x}{a} - \frac{a}{x}$$

51. For the curve $f(x) = (x^2 + 6x - 5)(1-x)$, find the slope of the tangent at $x = 3$.

LONG ANSWER TYPE QUESTIONS (6 MARKS)

Differentiate the following functions with respect to x from first principle:

52. $\frac{\cos x}{x}$

53. $x^2 \sin x$

Evaluate the following limits :

54. $\lim_{x \rightarrow \frac{\pi}{6}} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

55. $\lim_{x \rightarrow 0} \frac{\cos 2x - \cos 3x}{\cos 4x - 1}$

1. $\frac{1}{2}$

2. 3

3. 9

4. -5

5. $\frac{1}{2} - \frac{2}{x^2}$

6. $2x \tan x + x^2 \sec^2 x$

7. $\csc x - x \cot x \csc x$

8. 0

9. $2^x \log_e 2$

10. 1

11. $\frac{9}{2}$

14. $\lambda = -1$

15. $\frac{1}{2}$

16. $\frac{1}{2\sqrt{2}}$

17. 1

18. $\frac{5}{2} a^{\frac{3}{7}}$

19. $\frac{5}{2} (a+2)^{\frac{3}{2}}$

20. $\frac{m^2}{n^2}$

21. $\frac{1}{2}$

22. 2

23. $\cos a$

24. $\sin^3 a$

25. $-\frac{3}{2}$

26. 2

27. 1

28. $\frac{1}{e}$

29. $-\frac{1}{3}$

30. $\frac{2}{3\sqrt{3}}$

31. $2 \cos 2$

32. $\frac{1}{\sqrt{2x+3}}$

33. $\frac{x^2 - 1}{x^2}$

34. e^x

35. $\frac{1}{x}$

36. $- \operatorname{cosec} x \cdot \cot x$

37. $- \operatorname{cosec}^2 x$

38. $a^x \log_e a$

39. $6 - \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$

40. $3x^2 + \frac{3}{2x^{5/2}} - \frac{9}{2}\sqrt{x}$

41. $3x^2 + \frac{1}{x^2} - 1 - \frac{3}{x^4}$

42. $\frac{x^2}{(x \sin x + \cos x)^2}$

43. $x^2 e^x (3 \sin x + x \sin x + x \cos x)$

44. $e^x x^{n-1} \{n \log_a x + \log a + x \log_a x\}$

45.
$$\frac{\left(e^x + \frac{1}{x}\right) \sin x - \left(e^x + \log x\right) \cos x}{\sin^2 x}$$

46. $\frac{2}{x(1 - \log x)^2}$

47. $e^x \left(1 + \frac{1}{x} + x + \log x\right)$

49. $\sec^2 x$

51. -46

52. $\frac{-(x \sin x + \cos x)}{x^2}$

53. $2x \sin x + x^2 \cos x$

54. -3

55. $-\frac{5}{16}$