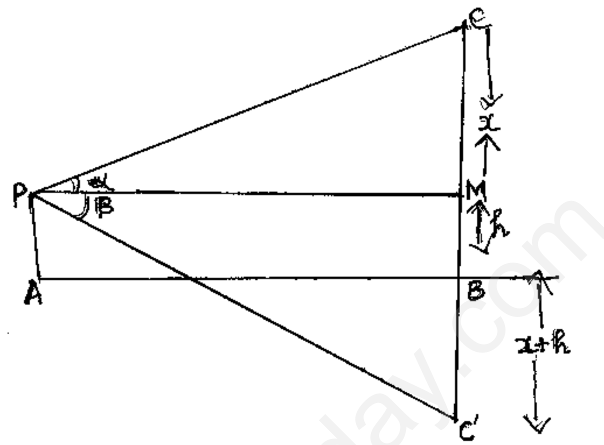
HEIGHTS AND DISTANCES

1. If the angle of elevation of cloud from a point 'h' meters above a lake is α and the angle of depression of its reflection in the lake is β, prove that the height of the cloud is .

Ans :



If the angle of elevation of cloud from a point 'n' meters above a lake is ∝ and the angle of depression of its reflection in the lake is β, prove that the height of the cloud is $h\left(\dfrac{\tan\beta+\tan\alpha}{\tan\beta-\tan\alpha}\right)$

Let AB be the surface of the lake and

Let p be an point of observation such that AP = h meters. Let c be the position of the cloud and c' be its reflection in the lake. Then ∠CPM = ∝ and ∠ MPC^1 = β.

Let CM = x.

Then, CB = CM + MB = CM + PA = x + h

In Δ CPM, we have $\tan \propto = \dfrac{CM}{PM}$

$\Rightarrow \quad \tan \propto = \dfrac{x}{AB}$

[∴ PM = AB]

$\Rightarrow$ AB = x cot∝ 1

In Δ PMC', we have

$\tan\beta = \dfrac{C'M}{PM}$

$\Rightarrow \quad \tan\beta = \dfrac{x+2h}{AB}$ [Θ C'M=C'B+BM = x + h + n]

$\Rightarrow$ AB = (x + 2h) cot β

From 1 & 2

x cot ∝ = (x + 2h) cot β

x (cot ∝ - cot β) = 2h cot β (on equating the values of AB)

$$\Rightarrow x\left(\frac{1}{\tan\alpha}-\frac{1}{\tan\beta}\right)=\frac{2h}{\tan\beta} \Rightarrow x\left(\frac{\tan\beta-\tan\alpha}{\tan\alpha+\tan\beta}\right)=\frac{2h}{\tan\beta}$$

$$\Rightarrow x=\frac{2h\tan\alpha}{\tan\beta-\tan\alpha}$$

Hence, the height CB of the cloud is given by CB is given by CB = x + h

$$\Rightarrow \text{CB}=\frac{2h\tan\alpha}{\tan\beta-\tan\alpha}+\text{h}$$

$$\Rightarrow \text{CB-}\frac{2h\tan\alpha+h\tan\beta-h\tan\alpha}{\tan\beta-\tan\alpha}=\frac{h(\tan\alpha+h\tan\beta)}{\tan\beta-\tan\alpha}$$

2. From an aero plane vertically above a straight horizontal road, the angles of depression of two consecutive milestones on opposite sides of the aero plane are observed to be α and β. Show that the height of the aero plane above the road is $\dfrac{\tan\alpha\ \tan\beta}{\tan\alpha+\tan\beta}$.

Ans:

Let P Q be h

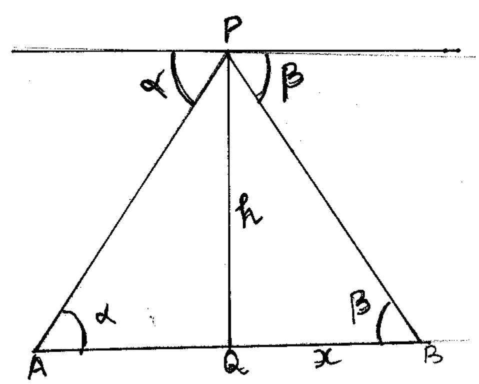
QB be x

Given : AB = 1 mile

QB = x

AQ = (1- x) mile

in ΔPAQ



Tan α = $\dfrac{PQ}{AQ}$

Tan α = $\dfrac{h}{1-x}$

$1-x=\dfrac{h}{Tan\alpha}$ ………….1

In ΔPQB

Tan β = $\dfrac{h}{x}$

x = $\dfrac{h}{Tan\beta}$

Substitute for x in equation (1)

$1=\dfrac{h}{Tan\beta}+\dfrac{h}{Tan\alpha}$

$$1 = h\left\{\frac{1}{Tan\beta} + \frac{1}{Tan\alpha}\right\}$$

$$\frac{1}{h} = \frac{Tan\beta + Tan\alpha}{Tan\beta Tan\alpha}$$

$$\therefore \text{h} = \frac{\tan\alpha \tan\beta}{\tan\alpha + \tan\beta}$$

3. Two stations due south of a tower, which leans towards north are at distances 'a' and 'b' from its foot. If α and β be the elevations of the top of the tower from the situation, prove that its inclination 'θ' to the horizontal given by

$$\cot\theta = \frac{b\cot\alpha - a\cot\beta}{b-a}$$

Ans: Let AB be the leaning tower and C and D be the given stations. Draw BL ⊥ DA produced.

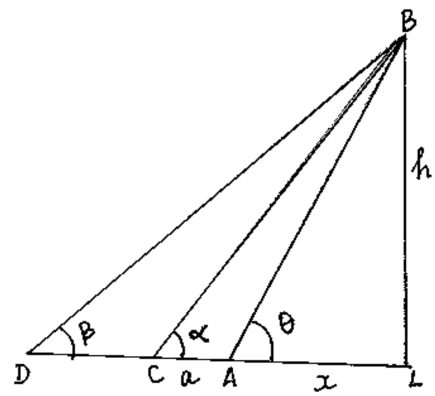
Then, ∠BAL = 0, ∠BCA = α, ∠BDC = a and DA = b.

Let AL = x and BL = h

In right ΔALB, we have :

$$\frac{AL}{BL} = \text{Cot } \theta \Rightarrow \frac{x}{h} = \text{Cot } \theta$$

$$\Rightarrow \frac{x}{h} = \text{Cot } \theta \Rightarrow x = \text{h cot } \theta \quad \text{.....(i)}$$



In right ΔBCL, we have :

$$\frac{CL}{BL} = \text{Cot } \alpha \Rightarrow \text{a + x = h cot } \alpha$$

$$\Rightarrow \text{a = h (cot } \alpha \text{ - cot } \theta)$$

$$\Rightarrow \text{h} = \frac{a}{(\cot\alpha - \cot\theta)} \quad \text{...(ii)}$$

In right ΔBDL, we have :

$$\frac{DL}{BL} = \cot\beta \Rightarrow \frac{DA + AL}{BL} = \cot\beta$$

$$\Rightarrow \frac{b+x}{h} = \cot\beta \Rightarrow \text{b + x = b cot } \beta$$

$$\Rightarrow \text{b = h ((cot } \beta \text{ - cot } \theta) \qquad \text{[using (i)]}$$

$$\Rightarrow \text{h} = \frac{b}{(\cot\beta - \cot\theta)} \quad \text{..........(iii)}$$

equating the value of h in (ii) and (iii), we get:

$$\frac{a}{(\cot\alpha - \cot\theta)} = \frac{b}{(\cot\beta - \cot\theta)}$$

$\Rightarrow$ a cot β - a cot θ = b cot α - b cot θ

$\Rightarrow$ (b – a) cot θ = b cot α - a cotβ

$\Rightarrow$ cot θ = $\dfrac{b\cot\alpha - a\cot\beta\theta}{(b-a)}$

4. The angle of elevation of the top of a tower from a point on the same level as the foot of the tower is α. On advancing ‘p’ meters towards the foot of the tower, the angle of elevation becomes β. show that the height ‘h’ of the tower is given by h= $\dfrac{p(\tan\alpha\ \tan\beta)}{\tan\beta - \tan\alpha}$

5. A boy standing on a horizontal plane finds a bird flying at a distance of 100m from him at an elevation of 30^0. A girl standing on the roof of 20 meter high building finds the angle of elevation of the same bird to be 45^0. Both the boy and the girl are on opposite sides of the bird. Find the distance of the bird from the girl. (Ans: 42.42m)

Ans: In right Δ ACB

Sin 30 = $\dfrac{AC}{AB}$

$\dfrac{1}{2} = \dfrac{AC}{100}$

2 AC = 100

AC = 50m

$\Rightarrow$ AF = (50 – 20) = 30m

In right Δ AFE

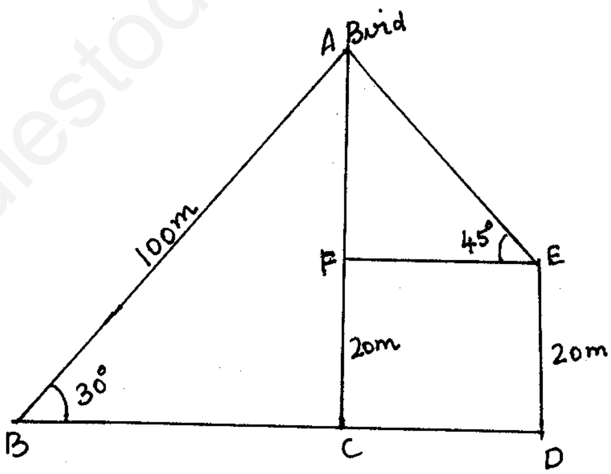
Sin 45 = $\dfrac{AF}{AE}$

$\Rightarrow \dfrac{1}{\sqrt{2}} = \dfrac{30}{AE}$

AE = 30 $\sqrt{2}$

= 30 x 1.414

= 42.42m

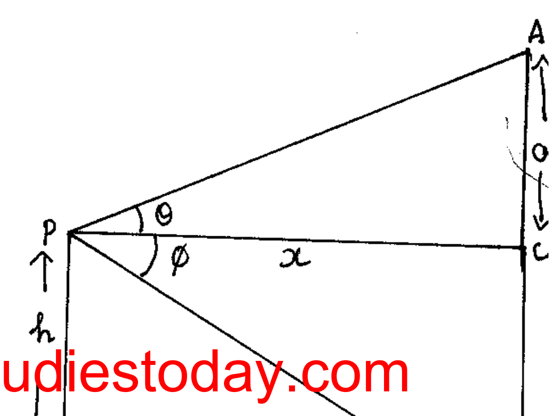


6. From a window x meters high above the ground in a street, the angles of elevation and depression of the top and the foot of the other house on the opposite side of the street are α and β respectively. Show that the height of the opposite house is $x(1 + \tan\alpha\ \cot\beta)$ Meters.

Ans: Let AB be the house and P be the window

Let BQ = x ∴ PC = x

Let AC = a



In Δ PQB, $\tan\theta = \frac{PQ}{QB}$ or $\tan\theta = \frac{h}{x}$

$\therefore$ $x = \frac{h}{\tan\theta} = h\cot\theta$

In Δ PAC, $\tan\theta = \frac{AC}{PC}$ or $\tan\theta = \frac{a}{x}$

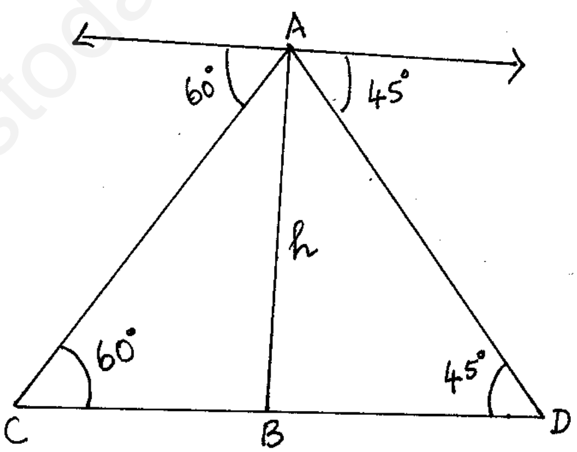
$\therefore$ $a = x\tan\theta >(h\cot\theta)\tan\theta = h\tan\theta\cot\theta$.

$\therefore$ the height of the tower = AB = AC + BC

$= a + h = h\tan\theta\cot\theta + h = h(\tan\theta\cot\theta + 1)$

7. Two ships are sailing in the sea on either side of a lighthouse; the angles of depression of two ships as observed from the top of the lighthouse are 60^0 and 45^0 respectively. If the distance between the ships is $200(\frac{1+\sqrt{3}}{\sqrt{3}})$ meters, find the height of the lighthouse. (Ans:200m)

Ans: In right Δ ABC



$\text{Tan } 60 = \frac{h}{BC}$

$\sqrt{3} = \frac{h}{BC}$

$\Rightarrow H = \sqrt{3}\,BC$

In right Δ ABD

$\text{Tan } 45 = \frac{h}{BD}$

$h = BD$

$BC + BD = 200\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)$

$BC + \sqrt{3}\,BC = 200\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)$

$\Rightarrow BC = \frac{200(1+\sqrt{3}}{\sqrt{3}\sqrt{(1+\sqrt{3})}}$

$\therefore$ $h = \sqrt{3}\,BC$

$= \sqrt{3}\,\frac{200}{\sqrt{3}}$

$= 200m$

$\therefore$ height of light house = 200m

8. A round balloon of radius 'a' subtends an angle θ at the eye of the observer while the angle of elevation of its centre is Φ. Prove that the height of the center of the balloon is a sin θ cosec Φ /2.

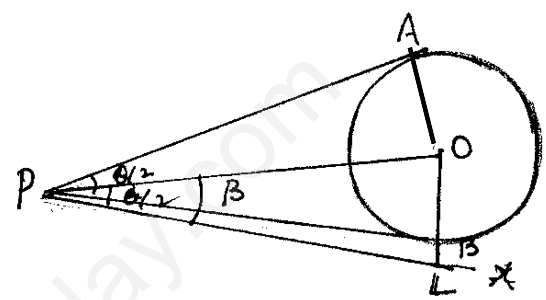
Ans: Let θ be the centre of the ballon of radius 'r' and 'p' the eye of the observer. Let PA, PB be tangents from P to ballong. Then $\angle APB = \theta$.

$\therefore \angle APO = \angle BPO = \frac{\theta}{2}$

Let OL be perpendicular from O on the horizontal PX. We are given that the angle of the elevation of the centre of the ballon is ϕ i.e.,

$\angle OPL = \phi$

In ΔOAP, we have $\sin \frac{\theta}{2} = \frac{OA}{OP}$



$\Rightarrow \sin \frac{\theta}{2} = \frac{a}{OP}$

$OP = a \operatorname{cosec} \frac{\theta}{2}$

In ΔOPL, we have $\sin\phi = \frac{OL}{OP}$

$\Rightarrow OL = OP \ \sin \phi = a \operatorname{cosec} \frac{\phi}{2} \sin \theta.$

Hence, the height of the center of the balloon is *a* sin θ cosec Φ /2.

9. The angle of elevation of a jet fighter from a point A on the ground is 60^0. After a flight of 15 seconds, the angle of elevation changes to 30^0. If the jet is flying at a speed of 720 km/hr, find the constant height at which the jet is flying.(Use $\sqrt{3}$ =1.732 (Ans: 2598m)

Ans:

36 km / hr = 10m / sec

720 km / h = $\frac{10 \times 720}{36}$

Speed = 200 m/s

Distance of jet from

AE = speed x time

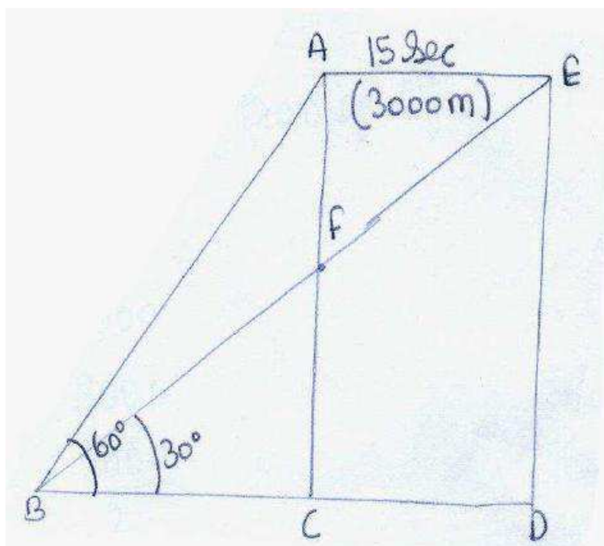
= 200 x 15

= 3000 m

$\tan 60^o = \frac{AC}{BC} \left(\frac{oppositeside}{adjacentside}\right)$

$\sqrt{3} = \frac{AC}{BC}$

BC $\sqrt{3}$ = AC



AC = ED (constant height)

$\therefore$ BC $\sqrt{3}$ = ED ………………1

$\tan 30^{\circ} = \dfrac{ED}{BC+CD}\left(\dfrac{oppositeside}{adjacentside}\right)$

$\dfrac{1}{\sqrt{3}} = \dfrac{ED}{BC+3000}$

$\dfrac{BC+3000}{\sqrt{3}}$ =ED

$\dfrac{BC+3000}{\sqrt{3}}$ =BC $\sqrt{3}$ (from equation 1)

BC + 3000 = 3BC

3BC – BC = 3000

2 BC = 3000

BC = $\dfrac{3000}{2}$

BC = 1500 m

ED = BC $\sqrt{3}$ (from equation 1)

= 1500 $\sqrt{3}$

= 1500 x 1.732

ED = 2598m

$\therefore$ The height of the jet fighter is 2598m.

10. A vertical post stands on a horizontal plane. The angle of elevation of the top is 60° and that of a point x metre be the height of the post, then prove that $x = \dfrac{2h}{3}$.

Self Practice

11. A fire in a building B is reported on telephone to two fire stations P and Q, 10km apart from each other on a straight road. P observes that the fire is at an angle of 60° to the road and Q observes that it is an angle of 45° to the road. Which station should send its team and how much will this team have to travel? (Ans:7.32km)

Self Practice

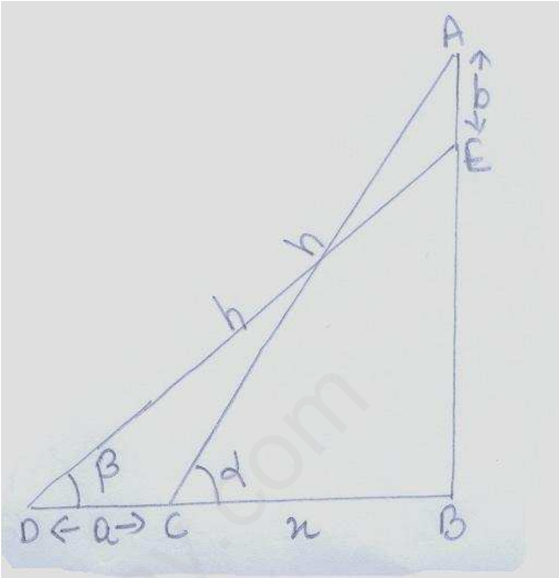
12. A ladder sets against a wall at an angle α to the horizontal. If the foot is pulled away from the wall through a distance of 'a', so that is slides a distance 'b' down the wall making an angle β with the horizontal. Show that $\dfrac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha} = \dfrac{a}{b}$.

Ans: Let CB = x m. Length of ladder remains same

$\text{Cos}\ \alpha = \frac{CB}{CA}$ $\quad \therefore$ ED = AC Let Ed be

$\text{Cos}\ \alpha = \frac{x}{h}$ $\quad \therefore$ ED = AC = h

x = hcos α ………(1)



$\cos\beta = \frac{DC + CB}{ED}$

$\cos\beta = \frac{a + x}{h}$

a + x = hcos β

x = hcos β – a ………(2)

from (1) & (2)

hcos α = hcos β - a

hcos α - hcos β = - a

-a = h(cosα - cosβ) ……...(3)

$\text{Sin}\ \alpha = \frac{AE + EB}{AC}$

$\text{Sin}\ \alpha = \frac{b + EB}{h}$

hSin α – b = EB

EB = hSin α – b ……...(4)

$\text{Sin}\ \beta = \frac{EB}{DE}$

$\text{Sin}\ \beta = \frac{EB}{h}$

EB = hSin β ………(5)

From (4) & (5)

hSin β = hSin α – b

b = hsin α – hSin β

-b = h(Sin β - Sin α) ……...(6)

Divide equation (3) with equation (6)

$$\frac{-a}{-b} = \frac{h(\cos\alpha - \cos\beta)}{h(\sin\beta - Sin\alpha)}$$

$$\therefore \frac{a}{b} = \frac{Cos\alpha - Cos\beta}{Sin\beta - Sin\alpha}$$

13. Two stations due south of a leaning tower which leans towards the north are at distances a and b from its foot. If α, β be the elevations of the top of the tower from these stations, prove that its inclination φ is given by $\cot\varphi = \dfrac{b\cot\alpha - a\cot\beta}{b-a}$.

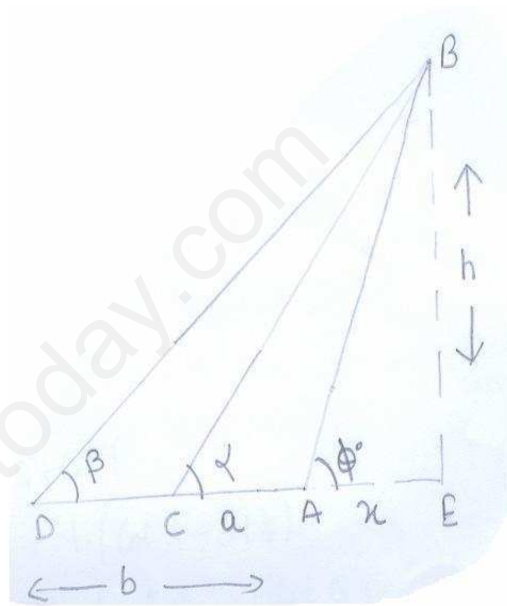
Ans:

Let AE = x, BE = h

Tan ϕ = $\dfrac{BE}{AE} = \dfrac{h}{x}$

x= h x $\dfrac{1}{\tan\phi}$

x = h cot ϕ ---------------1



tan α = $\dfrac{BE}{CE} = \dfrac{h}{a+x}$

a + x = h cot α

x= h cot α - a --------------2

tan β = $\dfrac{BE}{DE} = \dfrac{h}{b+x}$

b+x = h cot β

x = h cot β - b ---------------3

from 1 and 2

h cot ϕ = h cot α - a

h (cot ϕ + cot α) = a

h = $\dfrac{a}{-\cot\phi + \cot\alpha}$ -------------4

from 1 and 3

h cot ϕ = h cot β - b

h (cot ϕ - cot β) = b

h= $\dfrac{b}{-\cot\phi + \cot\beta}$

from 4 and 5

$$\frac{a}{-\cot\phi + \cot\alpha} = \frac{b}{-\cot\phi + \cot\beta}$$

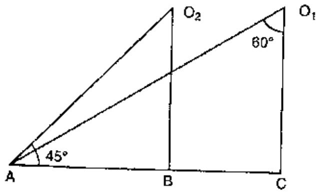
a (cot β - cot ϕ) = b (cot α - cot ϕ)

- a cot ϕ + b cot ϕ = b cot α - a cot β

(b – a) cot ϕ = b cot α - a cot β

$$\cot \phi = \frac{b \cot \alpha - a \cot \beta}{b-a}$$

14. In Figure, what are the angles of depression from the observing positions O_1 and O_2 of the object at A?



(Ans: 30^o ,45^o)

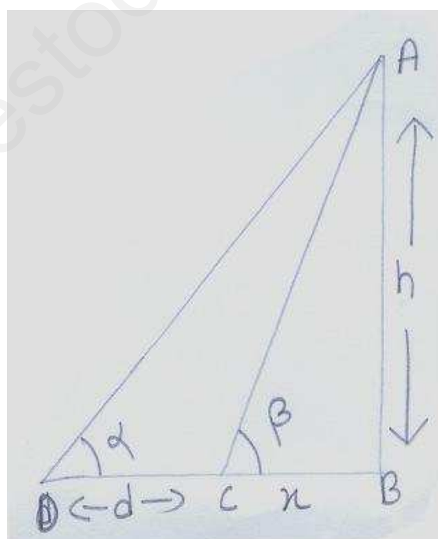
Self Practice

15. The angle of elevation of the top of a tower standing on a horizontal plane from a point A is α. After walking a distance d towards the foot of the tower the angle of elevation is found to be β. Find the height of the tower. (Ans: $\frac{d}{\cot \alpha - \cot \beta}$)

Ans:

Let BC = x

$\tan \beta = \frac{AB}{CB}$



$\tan \beta = \frac{h}{x}$

$x = \frac{h}{\tan \beta}$

$x = h \cot \beta$ ---------(1)

$\tan \alpha = \frac{AB}{DC+CB}$

$\tan \alpha = \frac{h}{d+x}$

$d + x = \frac{h}{\tan \alpha} = h\cot\alpha$

$x = h \cot \alpha - d$ ----------(2)

from 1 and 2

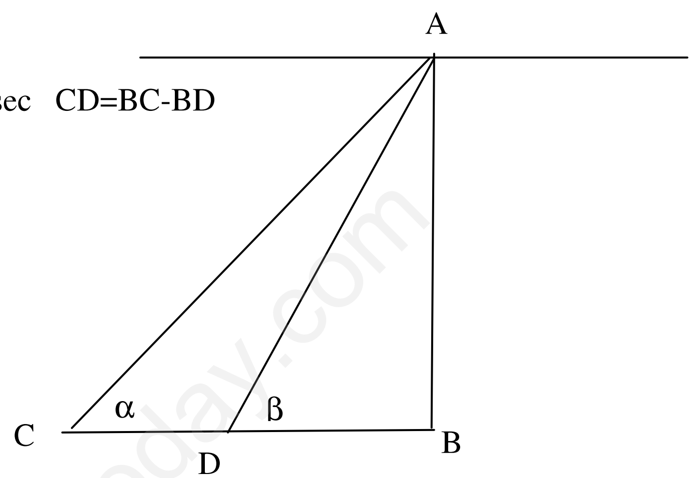
$h \cot \beta = h \cot \alpha - d$

$h (\cot \alpha - \cot \beta) = d$

$h = \frac{d}{\cot \alpha - \cot \beta}$

16. A man on a top of a tower observes a truck at an angle of depression α where $\tan\alpha = \frac{1}{\sqrt{5}}$ and sees that it is moving towards the base of the tower. Ten minutes later, the angle of depression of the truck is found to be β where $\tan\beta=\sqrt{5}$, if the truck is moving at a uniform speed, determine how much more time it will take to reach the base of the tower...

10 minutes=600sec



Ans: Let the speed of the truck be x m/sec CD=BC-BD

In right triangle ABC

$\tan\alpha = \frac{h}{BC}$ $\tan\alpha = \frac{1}{\sqrt{5}}$

$BC=h\sqrt{5}$............ 1

In right triangle ABD

$\tan\beta = \frac{h}{BD}$

$h=\sqrt{5}BD$ ($\tan\beta=\sqrt{5}$)

CD=BC-BD (CD=600x)

600x =5BD-BD

BD=150x

Time taken $= \frac{150x}{x}$

=150 seconds

Time taken by the truck to reach the tower is 150 sec.